Lecture 17:
Curves

Reading

- Hearn & Baker, 10.6 - 10.9

Optional

- Bartels, Beatty, and Barsky. *An Introduction to Splines for use in Computer Graphics and Geometric Modeling*, 1987.
- Farin. *Curves and Surfaces for CAGD: A Practical Guide*, 4th ed., 1997.

Curves before computers

The “loftsman’s spline”:

- long, narrow strip of wood or metal
- shaped by lead weights called “ducks”
- gives curves with second-order continuity, usually

Used for designing cars, ships, airplanes, etc.

But curves based on physical artifacts can’t be replicated well, since there’s no exact definition of what the curve is.

Around 1960, a lot of industrial designers were working on this problem.

Motivation for curves

What do we use curves for?

- building models

- movement paths

- animation

Mathematical curve representation

- Explicit $y=f(x)$
  - what if the curve isn't a function?
- Implicit $f(x,y,z) = 0$
  - hard to work with.
- Parametric $(f(u),g(u))$

Parametric polynomial curves

We'll use parametric curves where the functions are all polynomials in the parameter.

$$x(u) = \sum_{k=0}^{n} a_k u^k$$

$$y(u) = \sum_{k=0}^{n} b_k u^k$$

Advantages:

- easy (and efficient) to compute
- infinitely differentiable

Cubic curves

Fix n=3

For simplicity we define each cubic function within the range

$$0 \le t \le 1$$

$$\mathbf{Q}(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}$$

$$Q_x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$Q_y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$
$$Q_z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

Compact representation

Place all coefficients into a matrix

$$\mathbf{C} = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$$

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = \mathbf{T} \cdot \mathbf{C}$$

$$\frac{d}{dt} Q(t) = Q'(t) = \frac{d}{dt} \mathbf{T} \cdot \mathbf{C} = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \cdot \mathbf{C}$$

Controlling the cubic

Q: How many constraints do we need to specify to fully determine the cubic **Q**(t)?

Constraining the cubics

Redefine **C** as a product of the **basis matrix M** and the 4-element column vector of constraints or **geometry vector G**

$$\mathbf{C} = \mathbf{M} \cdot \mathbf{G}$$

$$\mathbf{Q}(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} G_{1x} & G_{1y} & G_{1z} \\ G_{2x} & G_{2y} & G_{2z} \\ G_{3x} & G_{3y} & G_{3z} \\ G_{4x} & G_{4y} & G_{4z} \end{bmatrix}$$

$$= \mathbf{T} \cdot \mathbf{M} \cdot \mathbf{G}$$

Hermite Curves

Determined by

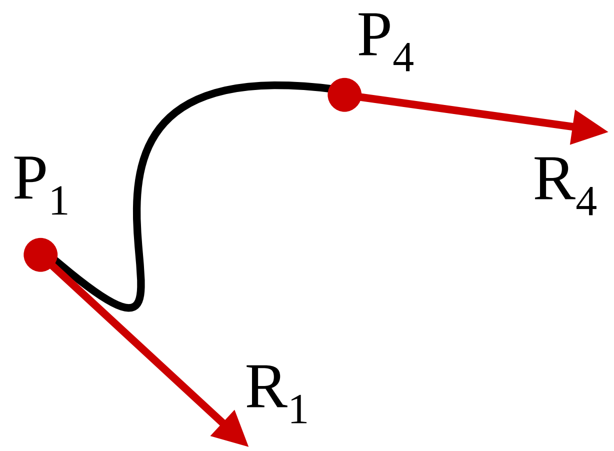
- endpoints P_1 and P_4
- tangent vectors at the endpoints R_1 and R_4

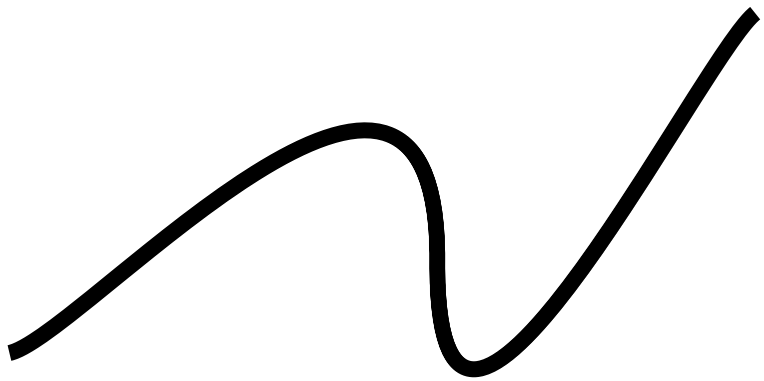
So

$$\mathbf{Q}(t) = \mathbf{T} \cdot \mathbf{M}_h \cdot \mathbf{G}_h$$

Where

$$\mathbf{G}_h = \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix}$$





Computing Hermite basis matrix

The constraints on Q(0) and Q(1) are found by direct substitution:

$$Q(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h$$

$$Q(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h$$

Tangents are defined by

$$Q'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h$$

so constraints on tangents are:

$$Q'(0) = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h$$

$$Q'(1) = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h$$

Computing Hermite basis matrix

Collecting all constraints we get

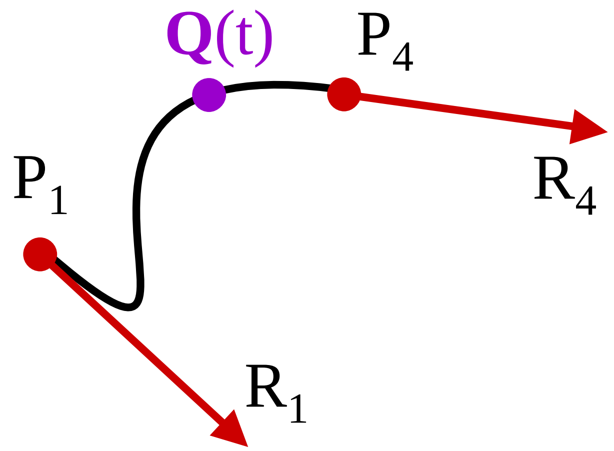
$$\begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix} = \mathbf{G}_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \mathbf{M}_h \cdot \mathbf{G}_h$$

So

$$\mathbf{M}_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Computing a point

Given two endpoints (P_1,P_4) and two endpoint tangent vectors (R_1 , R_4):



So

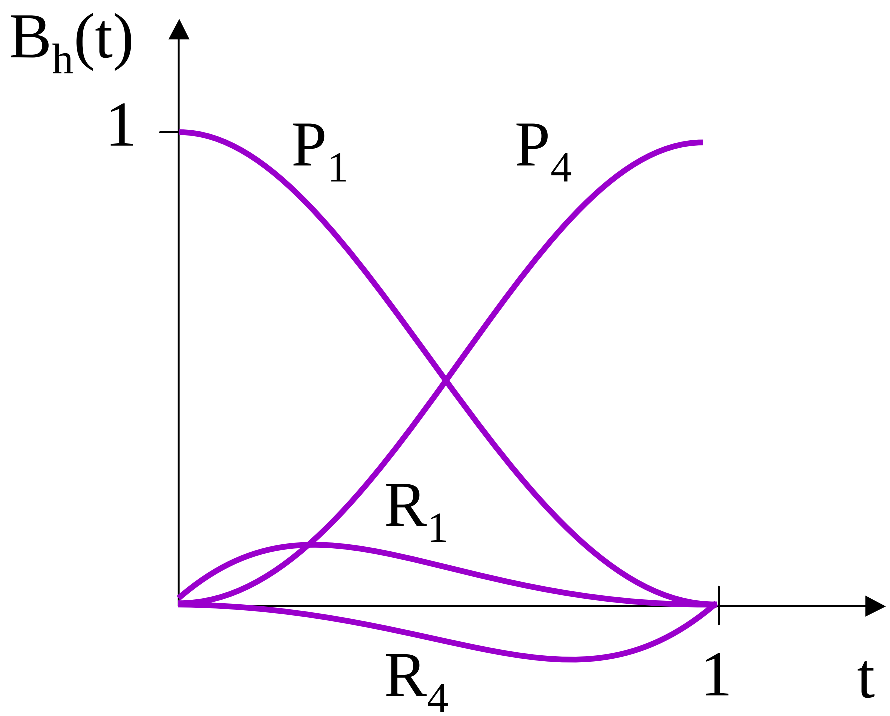
$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix}$$

Blending Functions

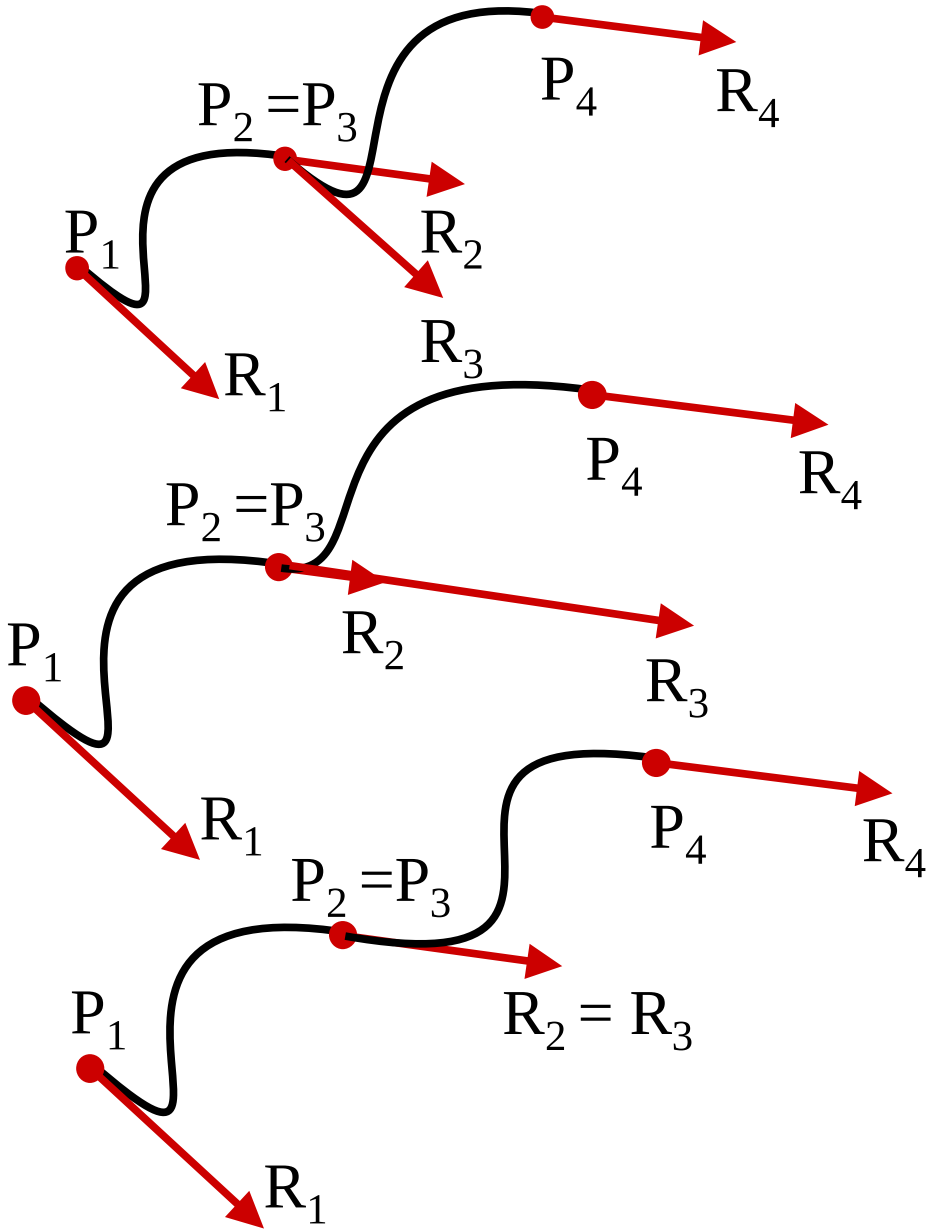
Polynomials weighting each element of the geometry vector

$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix}$$

$$= \mathbf{B}_h(t) \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix}$$



Continuity of Splines

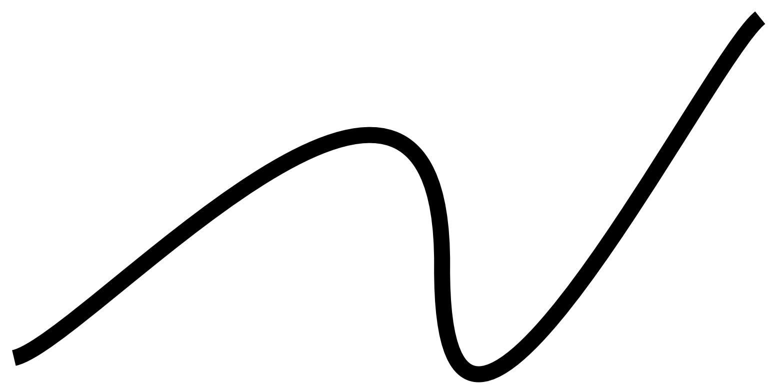


C^0: points coincide, velocities don't

G^1: points coincide, velocities have same direction

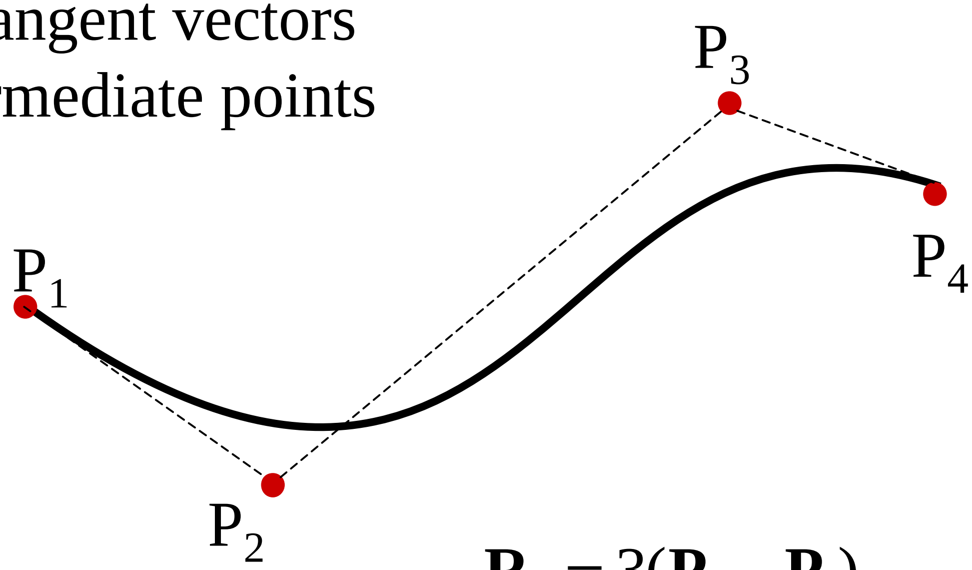
C^1: points and velocities coincide

Q: What's C^2?



Bezier Curves

Indirectly specify the tangent vectors by specifying two intermediate points



$$\mathbf{R}_1 = 3(\mathbf{P}_2 - \mathbf{P}_1)$$

$$\mathbf{R}_1 = 3(\mathbf{P}_4 - \mathbf{P}_3)$$

$$\mathbf{G}_b = \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{2x} & P_{2y} & P_{2z} \\ P_{3x} & P_{3y} & P_{3z} \\ P_{4x} & P_{4y} & P_{4z} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix}$$

Bezier basis matrix

Establish the relation between the Hermite and Besier geometry vectors:

$$\mathbf{G}_h = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix} = \mathbf{M}_{bh}\mathbf{G}_b$$

Bezier basis matrix

$$\mathbf{Q}(t) = \mathbf{T} \cdot \mathbf{M}_h \cdot \mathbf{G}_h = \mathbf{T} \cdot \mathbf{M}_h \cdot \left(\mathbf{M}_{hb} \cdot \mathbf{G}_b \right)$$

$$= \mathbf{T} \cdot \left(\mathbf{M}_h \cdot \mathbf{M}_{hb} \right) \cdot \mathbf{G}_b = \mathbf{T} \cdot \mathbf{M}_b \cdot \mathbf{G}_b$$

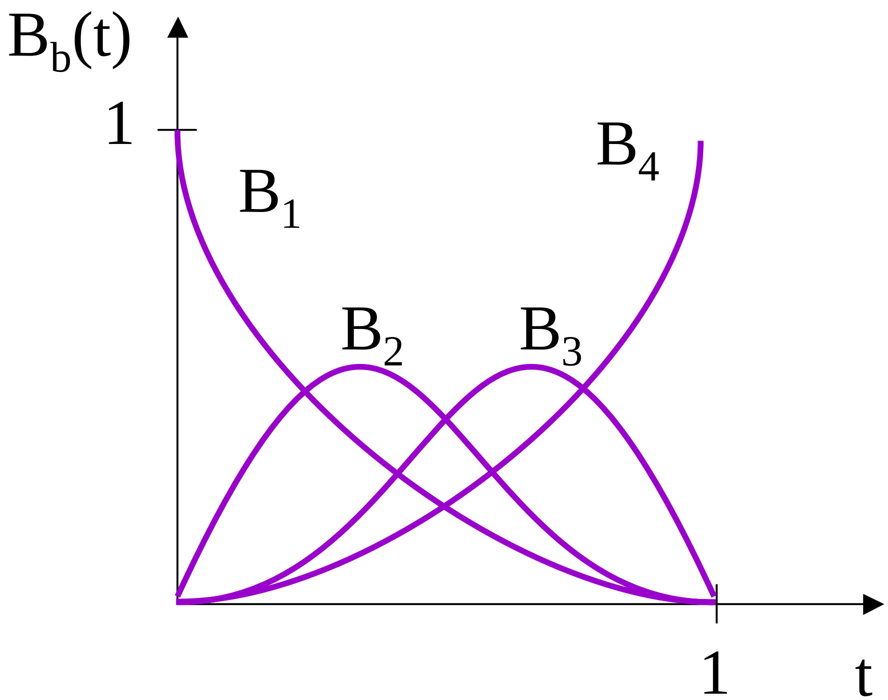
$$\mathbf{M}_b = \mathbf{M}_h \mathbf{M}_{hb} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}(t) = \mathbf{T} \cdot \mathbf{M}_b \cdot \mathbf{G}_b$$

Bezier Blending Functions

a.k.a. Bernstein polynomials

$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix} = \mathbf{B}_b(t) \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix}$$



Summary

• Use of parametric functions for curve modeling

• Enforcing constraints on cubic functions

• The meaning of basis matrix and geometry vector

•General procedure for computing the basis matrix

• Properties of Hermite and Bezier splines

• The meaning of blending functions

• Enforcing continuity across multiple curve segments