CSE 484 / CSE M 584 (Autumn 2011)

# Asymmetric Cryptography

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Thanks to Dan Boneh, Dieter Gollmann, John Manferdelli, John Mitchell, Vitaly Shmatikov, Bennet Yee, and many others for sample slides and materials ...

# Class updates

- Remember current events and security reviews are due **this Friday**
- *Lockpicks* and now **Fingerprint molds** are available in my office
  - Office hours or by appointment
- **Office hours** today in CSE 210

# Class updates (cont.)

- Lab 3 coming soon - **Privacy**
  - Working out the details with the lawyers
- Homework 3 (last homework!) out by Wednesday - Hashing and Asymmetric Cryptography

# Some Number Theory Facts

- Euler totient function $\varphi(n)$ where $n \geq 1$ is the number of integers in the $[1,n]$ interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
- Euler's theorem:

  if $a \in Z_n^*$, then $a^{\varphi(n)} = 1 \bmod n$

  $Z_n^*$: multiplicative group of integers mod n (integers relatively prime to n)
- Special case: <u>Fermat's Little Theorem</u>

  if p is prime and $\gcd(a,p)=1$, then $a^{p-1} = 1 \bmod p$

# RSA Cryptosystem

[Rivest, Shamir, Adleman 1977]

- **Key generation:**
  - Generate large primes p, q
    - Say, 1024 bits each (need primality testing, too)
  - Compute $n=pq$ and $\varphi(n)=(p-1)(q-1)$
  - Choose small e, relatively prime to $\varphi(n)$
    - Typically, $e=3$ or $e=2^{16}+1=65537$ (why?)
  - Compute unique d such that $ed = 1 \bmod \varphi(n)$
  - Public key = (e,n); private key = (d,n)
- **Encryption** of m: $c = m^e \bmod n$
  - Modular exponentiation by repeated squaring
- **Decryption** of c: $c^d \bmod n = (m^e)^d \bmod n = m$

# Why RSA Decryption Works

- $e \cdot d = 1 \bmod \varphi(n)$, thus $e \cdot d = 1 + k \cdot \varphi(n)$ for some $k$
  Can rewrite: $e \cdot d = 1 + k(p-1)(q-1)$

- Let m be any integer in $Z_n$
- If $\gcd(m,p)=1$, then $m^{ed} = m \bmod p$
  - By Fermat's Little Theorem, $m^{p-1} = 1 \bmod p$
  - Raise both sides to the power $k(q-1)$ and multiply by m
  - $m^{1+k(p-1)(q-1)} = m \bmod p$, thus $m^{ed} = m \bmod p$
  - By the same argument, $m^{ed} = m \bmod q$
- Since p and q are distinct primes and $p \cdot q = n$,
  $m^{ed} = m \bmod n$ (using the Chinese Remainder Theorem)
- True for all m in $Z_n$, not just m in $Z_n^*$

# Why Is RSA Secure?

- RSA problem: given $n=pq$, $e$ such that $\gcd(e,(p-1)(q-1))=1$ and $c$, find $m$ such that $m^e=c \bmod n$
  - i.e., recover m from ciphertext c and public key (n,e) by taking $e^{th}$ root of c
  - There is no known efficient algorithm for doing this
- Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e1}p_2^{e2}\ldots p_k^{ek}$
- If factoring is easy, then RSA problem is easy, but there is no known reduction from factoring to RSA
  - It may be possible to break RSA without factoring n

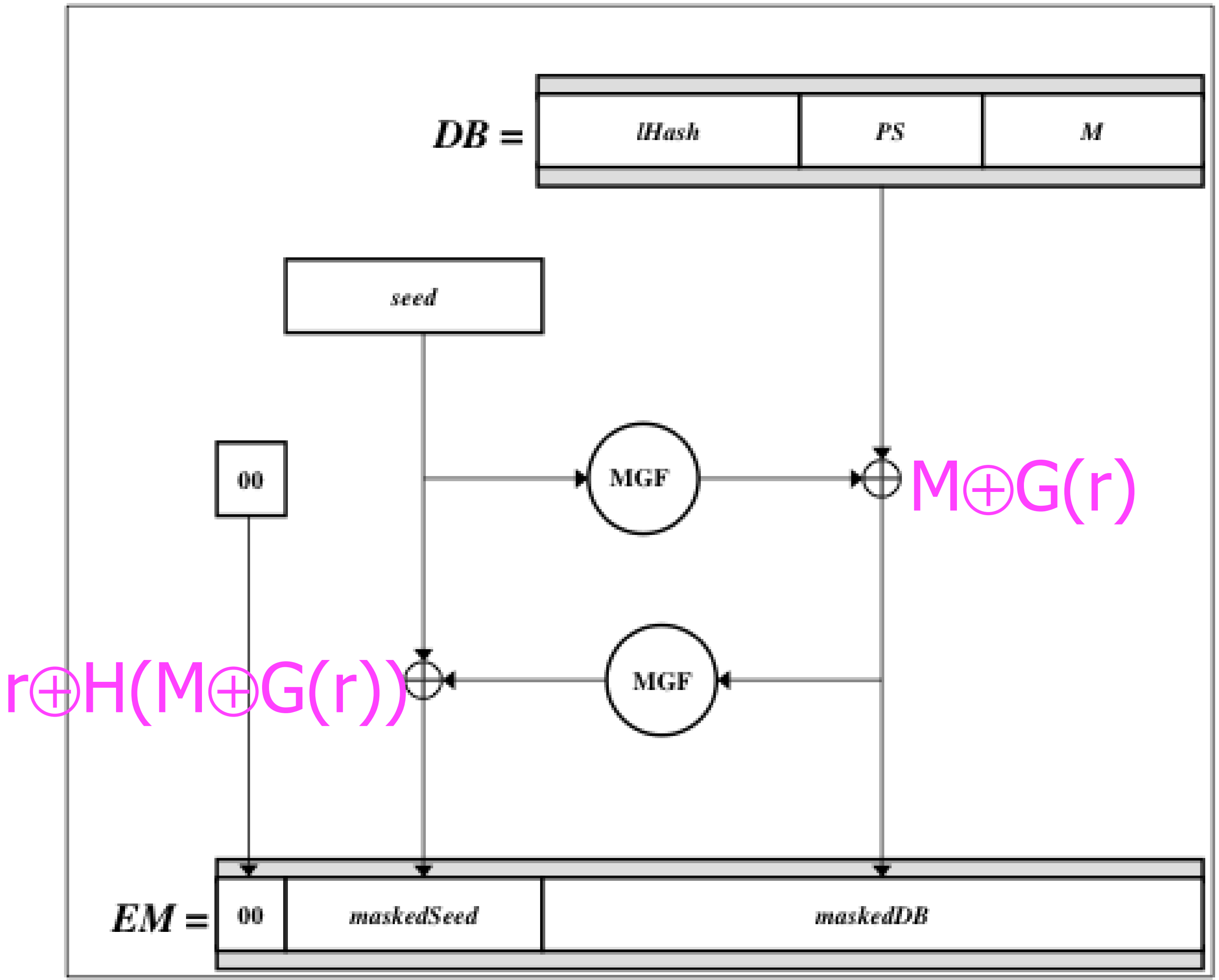
# Caveats

- $e = 3$ is a common exponent
  - If $m < n^{1/3}$, then $c = m^3 < n$ and can just take the cube root of c to recover m
    - Even problems if "pad" m in some ways [Hastad]
  - Let $c_i = m^3 \bmod n_i$ - same message is encrypted to three people
    - Adversary can compute $m^3 \bmod n_1 n_2 n_3$ (using CRT)
    - Then take ordinary cube root to recover m
- Don't use RSA directly for privacy!

# Integrity in RSA Encryption

- Plain RSA does not provide integrity
  - Given encryptions of $m_1$ and $m_2$, attacker can create encryption of $m_1 \cdot m_2$
    - $(m_1^e) \cdot (m_2^e) \bmod n = (m_1 \cdot m_2)^e \bmod n$
  - Attacker can convert m into $m^k$ without decrypting
    - $(m_1^e)^k \bmod n = (m^k)^e \bmod n$
- In practice, OAEP is used: instead of encrypting M, encrypt $M \oplus G(r) \,;\, r \oplus H(M \oplus G(r))$
  - r is random and fresh, G and H are hash functions
  - Resulting encryption is plaintext-aware: infeasible to compute a valid encryption without knowing plaintext
    - … if hash functions are "good" and RSA problem is hard

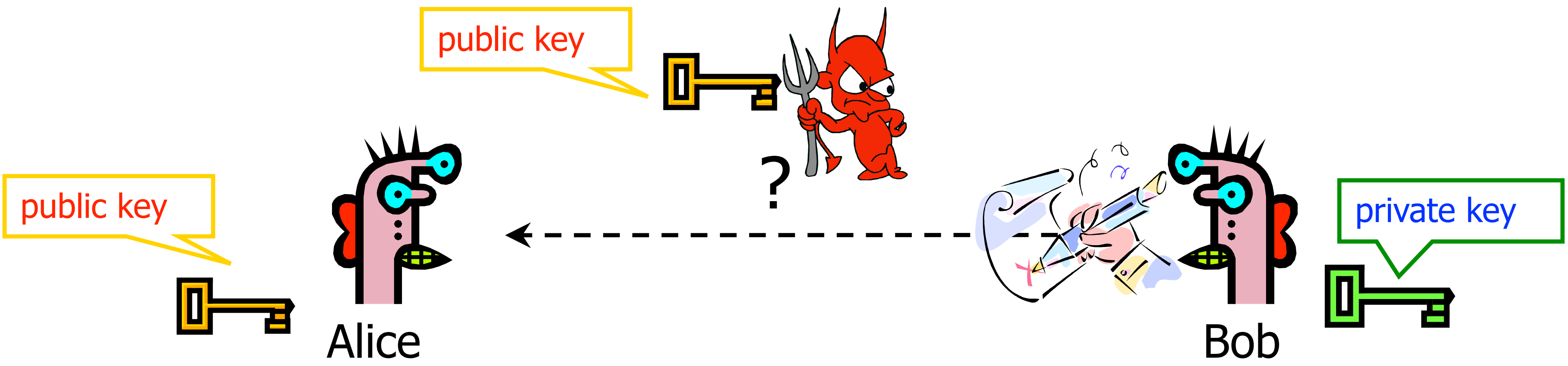
# OAEP (image from PKCS #1 v2.1)



# Today So Far

- Defined RSA primitives
  - Encryption and Decryption
  - Underlying number theory
  - Practical concerns, some mis-uses
  - OAEP

# Digital Signatures: Basic Idea




<u>Given</u>: Everybody knows Bob's public key

Only Bob knows the corresponding private key

<u>Goal</u>: Bob sends a "digitally signed" message

1. To compute a signature, must know the private key
2. To verify a signature, enough to know the public key

# RSA Signatures

- Public key is (n,e), private key is d
- To sign message m: $s = m^d \bmod n$
  - Signing and decryption are the same **underlying** operation in RSA
  - It's infeasible to compute s on m if you don't know d
- To verify signature s on message m:

  $s^e \bmod n = (m^d)^e \bmod n = m$
  - Just like encryption
  - Anyone who knows n and e (public key) can verify signatures produced with d (private key)
- In practice, also need padding & hashing
  - Standard padding/hashing schemes exist for RSA signatures

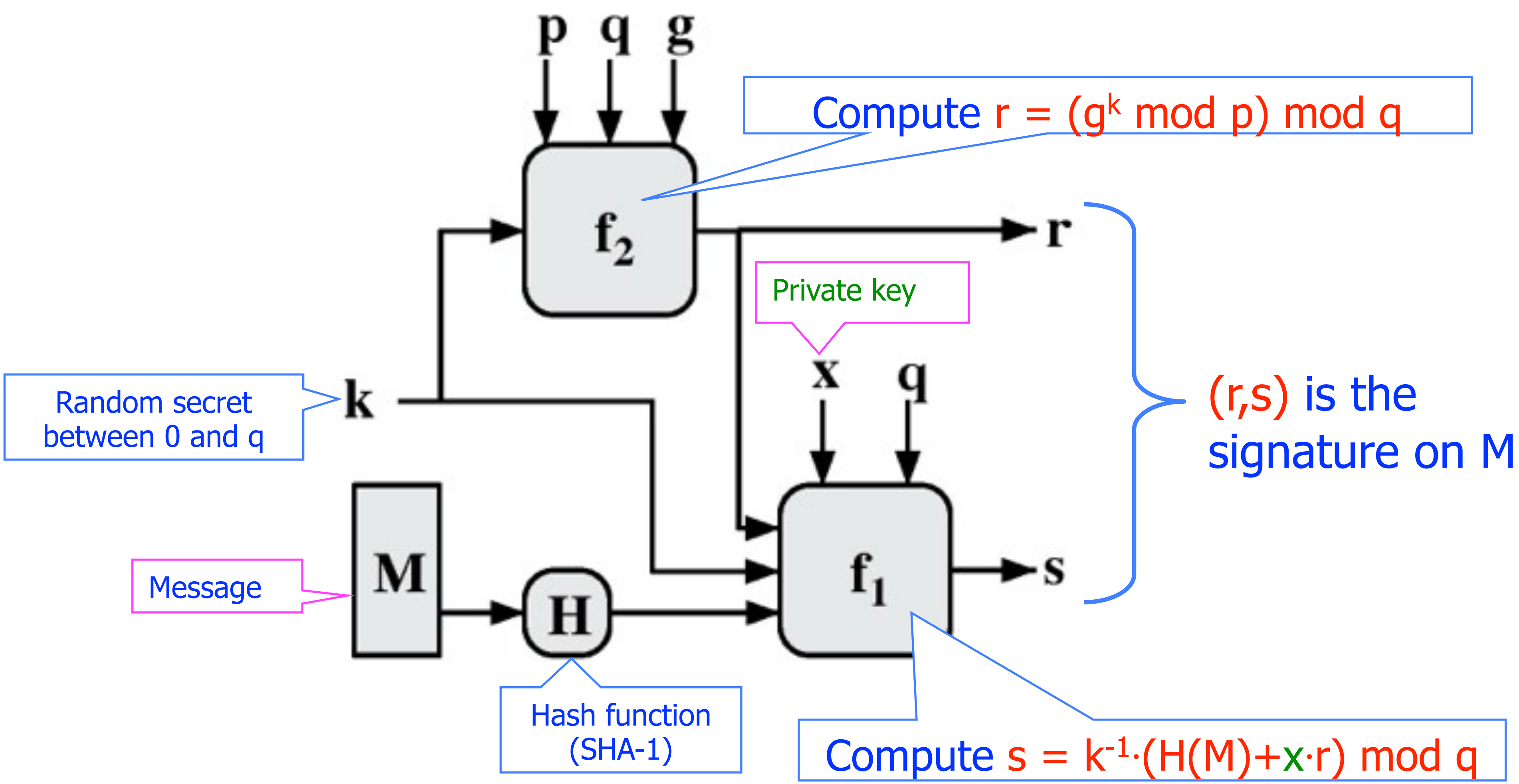
# Encryption and Signatures

- Often people think:  Encryption and decryption are inverses.
- That's a common view
  - True for the RSA **primitive (underlying component)**
- But not one we'll take
  - To really use RSA, we need padding
  - And there are many other decryption methods

# Digital Signature Standard (DSS)

- U.S. government standard (1991-94)
  - Modification of the ElGamal signature scheme (1985)
- Key generation:
  - Generate large primes p, q such that q divides p-1
    - $2^{159} < q < 2^{160}$, $2^{511+64t} < p < 2^{512+64t}$ where $0 \le t \le 8$
  - Select $h \in Z_p^*$ and compute $g = h^{(p-1)/q} \bmod p$
  - Select random x such $1 \le x \le q-1$, compute $y = g^x \bmod p$
- Public key: $(p, q, g, y = g^x \bmod p)$, private key: $x$
- Security of DSS requires hardness of discrete log
  - If could solve discrete logarithm problem, would extract x (private key) from $g^x \bmod p$ (public key)

# DSS: Signing a Message (Skim)



# DSS: Verifying a Signature (Skim)