CSE 484 / CSE M 584 (Spring 2012)

Asymmetric Cryptography

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Goals for Today

- Asymmetric Cryptography

- Lab 3 this week
- HW 2 also announced this week.
- There will also be an extra credit HW assignment (can only help your grade)

Requirements for Public-Key Encryption

- **Key generation:** computationally easy to generate a pair (public key PK, private key SK)
  - Computationally infeasible to determine private key SK given only public key PK
- **Encryption:** given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- **Decryption:** given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to compute M from C without SK
  - Even infeasible to learn partial information about M
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

Some Number Theory Facts

- Euler totient function $\varphi(n)$ where $n \geq 1$ is the number of integers in the $[1,n]$ interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
- Euler's theorem:

  if $a \in Z_n^*$, then $a^{\varphi(n)} = 1 \bmod n$

  Z_n^*: multiplicative group of integers mod n (integers relatively prime to n)
- Special case: <u>Fermat's Little Theorem</u>

  if p is prime and $\gcd(a,p)=1$, then $a^{p-1} = 1 \bmod p$

RSA Cryptosystem

[Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q
    - Say, 1024 bits each (need primality testing, too)
  - Compute $n=pq$ and $\varphi(n)=(p-1)(q-1)$
  - Choose small e, relatively prime to $\varphi(n)$
    - Typically, $e=3$ or $e=2^{16}+1=65537$ (why?)
  - Compute unique d such that $ed = 1 \bmod \varphi(n)$
  - Public key = (e,n); private key = (d,n)
- Encryption of m: $c = m^e \bmod n$
  - Modular exponentiation by repeated squaring
- Decryption of c: $c^d \bmod n = (m^e)^d \bmod n = m$

Why RSA Decryption Works

- $e \cdot d = 1 \bmod \varphi(n)$, thus $e \cdot d = 1 + k \cdot \varphi(n)$ for some k
  Can rewrite: $e \cdot d = 1 + k(p-1)(q-1)$

- Let m be any integer in Z_n
- If $\gcd(m,p)=1$, then $m^{ed} = m \bmod p$
  - By Fermat's Little Theorem, $m^{p-1} = 1 \bmod p$
  - Raise both sides to the power $k(q-1)$ and multiply by m
  - $m^{1+k(p-1)(q-1)} = m \bmod p$, thus $m^{ed} = m \bmod p$
  - By the same argument, $m^{ed} = m \bmod q$
- Since p and q are distinct primes and $p \cdot q = n$,
  $m^{ed} = m \bmod n$ (using the Chinese Remainder Theorem)
- True for all m in Z_n, not just m in Z_n^*

Why Is RSA Secure?

- RSA problem: given $n=pq$, e such that $\gcd(e,(p-1)(q-1))=1$ and c, find m such that $m^e=c \bmod n$
  - i.e., recover m from ciphertext c and public key (n,e) by taking e^{th} root of c
  - There is no known efficient algorithm for doing this
- Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$
- If factoring is easy, then RSA problem is easy (because knowing factors means you can compute d), but there is no known reduction from factoring to RSA
  - It may be possible to break RSA without factoring n -- but if it is, we don't know how

On RSA encryption

- Encrypted message needs to be in interpreted as an integer less than n
  - Reason: Otherwise can't decrypt.
  - Message is very often a symmetric encryption key.
- But still not quite that simple

Caveats

- e =3 is a common exponent
  - If $m < n^{1/3}$, then $c = m^3 < n$ and can just take the cube root of c to recover m (i.e., no operations taken module n)
    - Even problems if "pad" m in some ways [Hastad]
  - Let $c_i = m^3 \bmod n_i$ - same message is encrypted to three people
    - Adversary can compute $m^3 \bmod n_1 n_2 n_3$ (using CRT)
    - Then take ordinary cube root to recover m
- Don't use RSA **directly** for privacy! Need to pre-process input in some way.

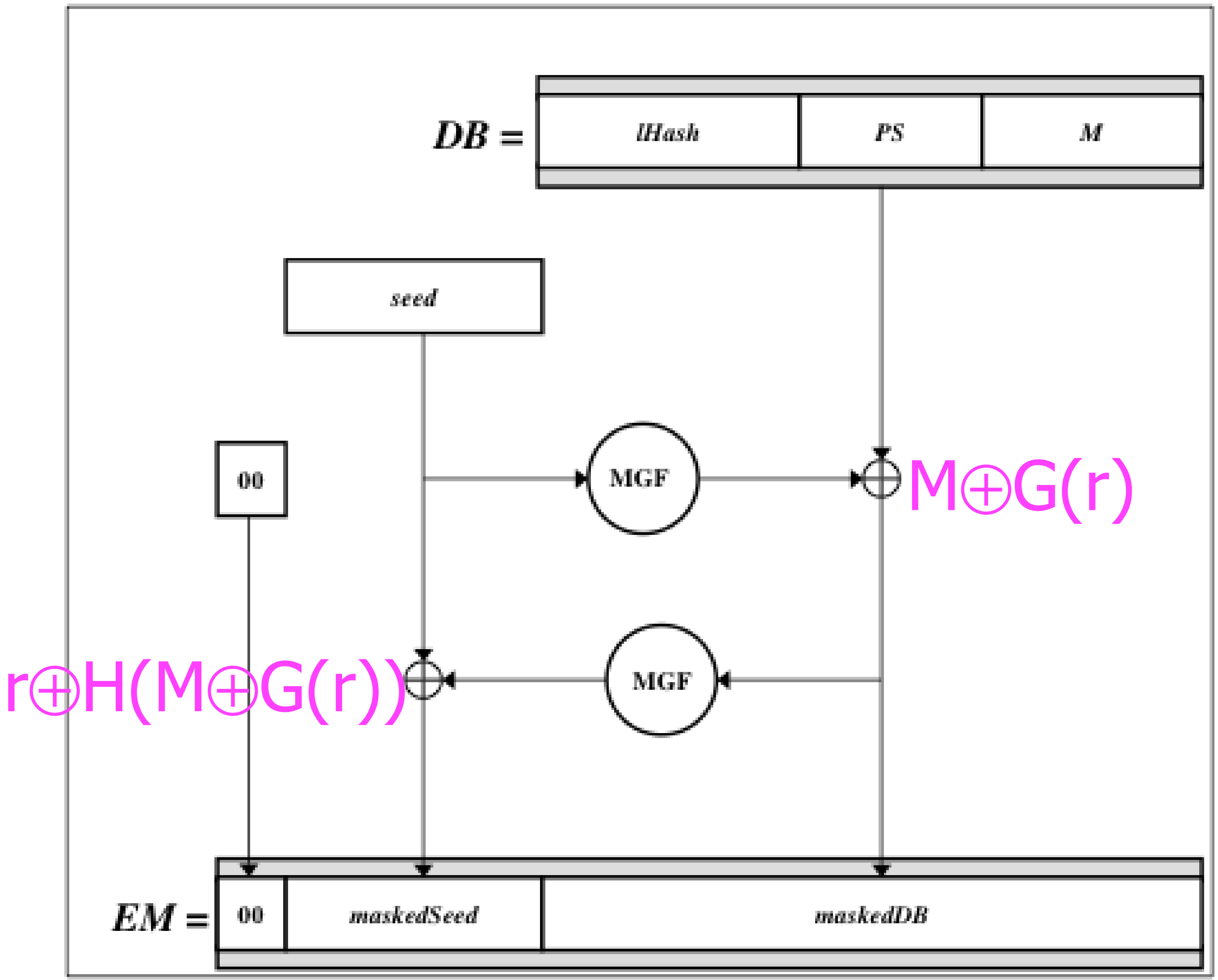
Sample Encryption

- 26 2 15 13 7 14 13 13 1 28 14 15 13 14 20 9 6 31 25 26 14 16 23 15 26 2 6 13 1
- P=3, Q=11, N=33, E=7, D=3
- 'A' converted to 1 before encryption; 'B' Converted to 2 before encryption; ...
- A-1 B-2 C-3 D-4 E-5 F-6 G-7 H-8 I-9 J-10 K-11 L-12 M-13 N-14 O-15 P-16 Q-17 R-18 S-19 T-20 U-21 V-22 W-23 X-24 Y-25 Z-26
- http://www.wolframalpha.com/

Integrity in RSA Encryption

- Plain RSA does not provide integrity
  - Given encryptions of m_1 and m_2, attacker can create encryption of $m_1 \cdot m_2$
    - $(m_1^e) \cdot (m_2^e) \bmod n = (m_1 \cdot m_2)^e \bmod n$
  - Attacker can convert m into m^k without decrypting
    - $(m_1^e)^k \bmod n = (m^k)^e \bmod n$
- In practice, OAEP is used: instead of encrypting M, encrypt $M \oplus G(r)\ ;\ r \oplus H(M \oplus G(r))$
  - r is random and fresh, G and H are hash functions
  - Resulting encryption is plaintext-aware: infeasible to compute a valid encryption without knowing plaintext
    - … if hash functions are "good" and RSA problem is hard

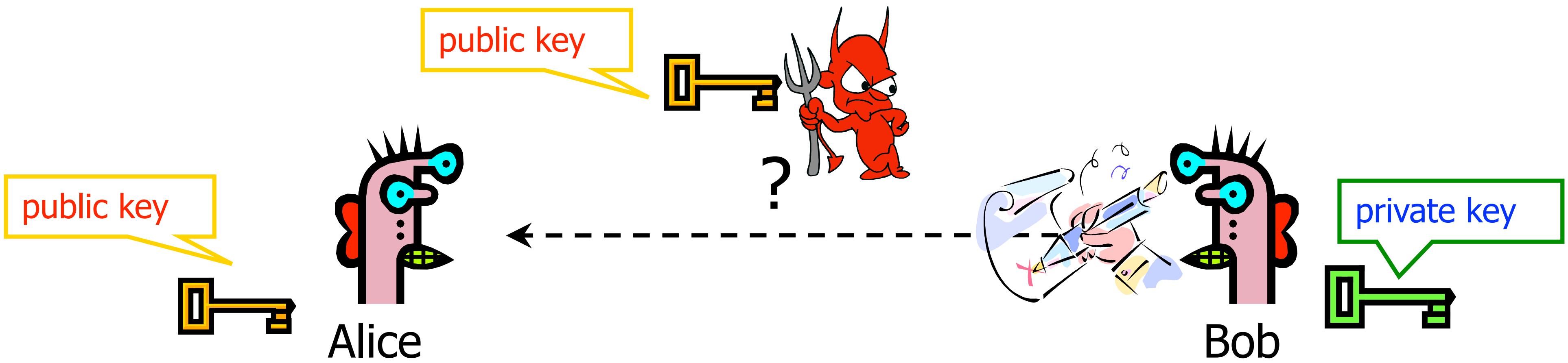
OAEP (image from PKCS #1 v2.1)



Summary of RSA

- Defined RSA primitives
  - Encryption and Decryption
  - Underlying number theory
  - Practical concerns, some mis-uses
  - OAEP

Digital Signatures: Basic Idea



<u>Given</u>: Everybody knows Bob's public key

Only Bob knows the corresponding private key

<u>Goal</u>: Bob sends a "digitally signed" message

1. To compute a signature, must know the private key
2. To verify a signature, enough to know the public key

RSA Signatures

- Public key is (n,e), private key is d
- To sign message m: $s = m^d \bmod n$
  - Signing and decryption are the same **underlying** operation in RSA
  - It's infeasible to compute s on m if you don't know d
- To verify signature s on message m:

  $s^e \bmod n = (m^d)^e \bmod n = m$
  - Just like encryption
  - Anyone who knows n and e (public key) can verify signatures produced with d (private key)
- In practice, also need padding & hashing
  - Standard padding/hashing schemes exist for RSA signatures

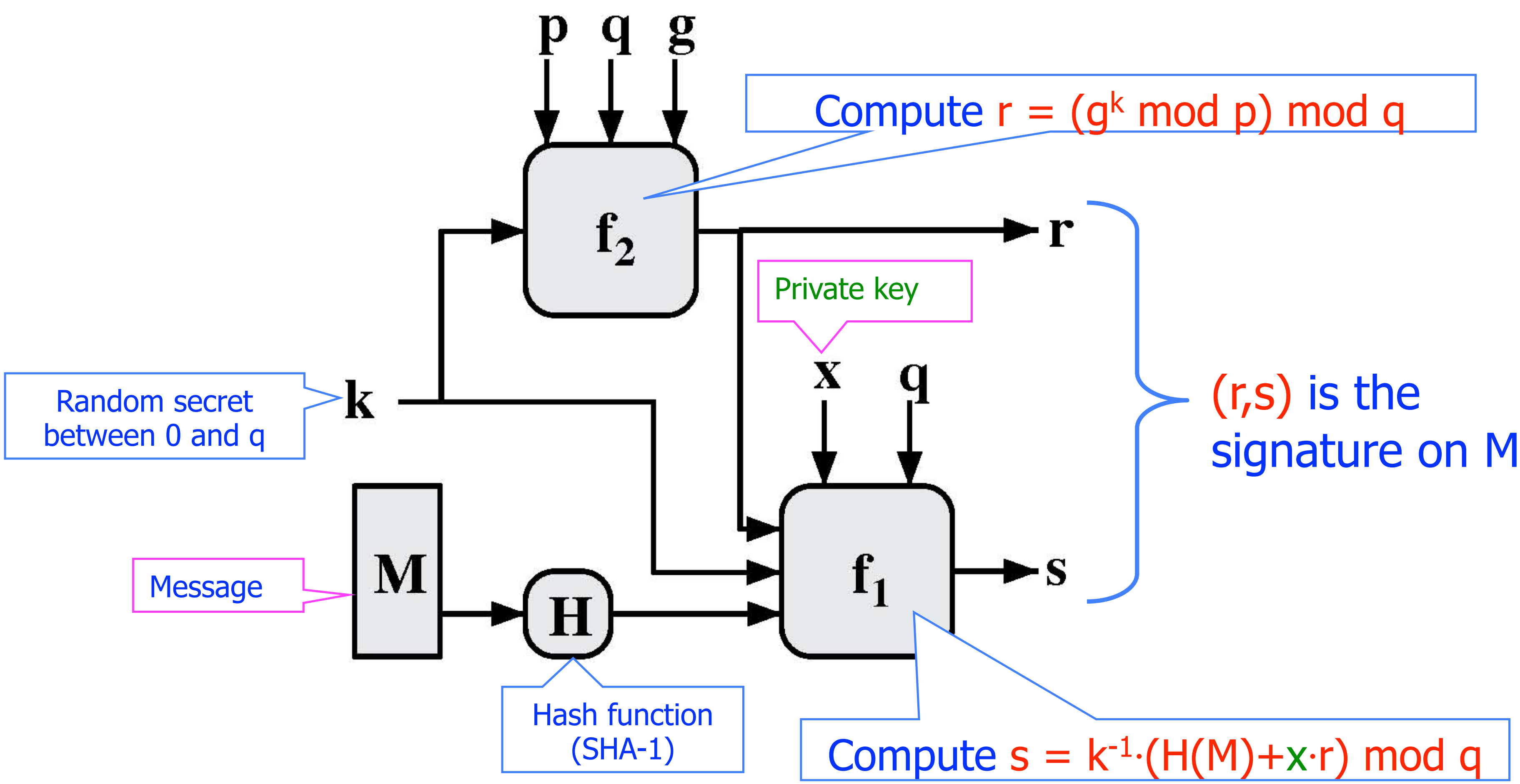
Encryption and Signatures

- Often people think: Encryption and decryption are inverses.
- That’s a common view
  - True for the RSA **primitive (underlying component)**
- But not one we’ll take
  - To really use RSA, we need padding
  - And there are many other decryption methods
  - And there are many other signing methods

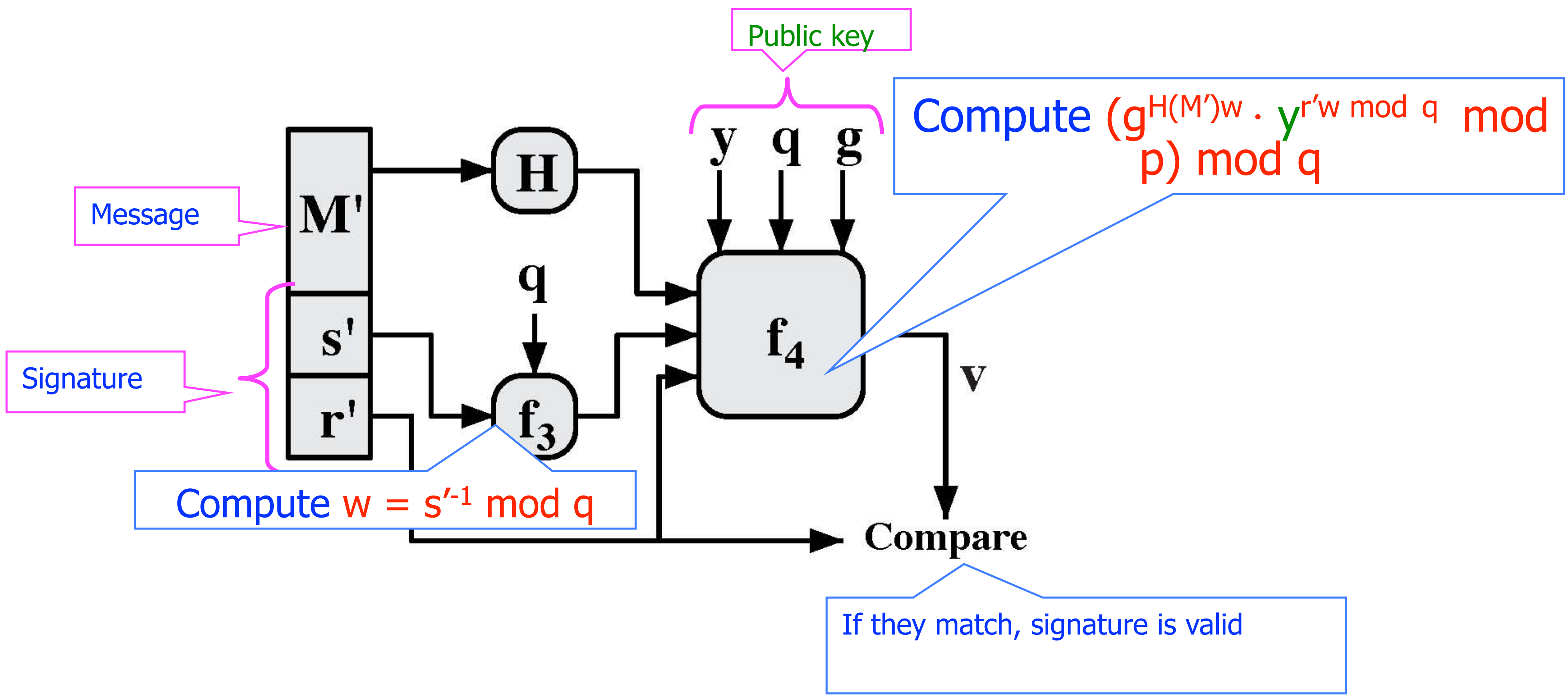
Digital Signature Standard (DSS)

- U.S. government standard (1991-94)
  - Modification of the ElGamal signature scheme (1985)
- Key generation:
  - Generate large primes p, q such that q divides p-1
    - $2^{159} < q < 2^{160}$, $2^{511+64t} < p < 2^{512+64t}$ where $0 \leq t \leq 8$
  - Select $h \in Z_p^*$ and compute $g = h^{(p-1)/q} \bmod p$
  - Select random x such $1 \leq x \leq q-1$, compute $y = g^x \bmod p$
- Public key: $(p, q, g, y = g^x \bmod p)$, private key: x
- Security of DSS requires hardness of discrete log
  - If could solve discrete logarithm problem, would extract x (private key) from $g^x \bmod p$ (public key)

DSS: Signing a Message (Skim)



DSS: Verifying a Signature (Skim)



Advantages of Public-Key Crypto

- Confidentiality without shared secrets
  - Very useful in open environments
  - No "chicken-and-egg" key establishment problem
    - With symmetric crypto, two parties must share a secret before they can exchange secret messages
    - Caveats to come
- Authentication without shared secrets
  - Use digital signatures to prove the origin of messages
- Reduce protection of information to protection of authenticity of public keys
  - No need to keep public keys secret, but must be sure that Alice's public key is <u>really</u> her true public key

Disadvantages of Public-Key Crypto

- Calculations are 2-3 orders of magnitude slower
  - Modular exponentiation is an expensive computation
  - Typical usage: use public-key cryptography to establish a shared secret, then switch to symmetric crypto
    - E.g., IPsec, SSL, SSH, ...
- Keys are longer
  - 1024+ bits (RSA) rather than 128 bits (AES)
- Relies on unproven number-theoretic assumptions
  - What if factoring is easy?
    - Factoring is believed to be neither P, nor NP-complete
  - (Of course, symmetric crypto also rests on unproven assumptions)

Note: Optimizing Exponentiation

- How to compute M^x mod N? Say x=13
- Sums of power of 2, x = 8+4+1 = $2^3+2^2+2^0$
- Can also write x in binary, e.g., x = 1101
- Can solve by repeated squaring
  - y = 1;
  - $y = y^2 * M \bmod N$ // $y = M$
  - $y = y^2 * M \bmod N$ // $y = M^2 * M = M^{2+1} = M^3$
  - $y = y^2 \bmod N$ // $y = (M^{2+1})^2 = M^{4+2}$
  - $y = y^2 * M \bmod N$ // $y = (M^{4+2})^2 * M = M^{8+4+1}$
- Does anyone see a potential issue?

Timing attacks

Collect timings for exponentiation with a bunch of messages M1, M2, ... (e.g., RSA signing operations with a private exponent)

Assume (inductively) know b_3=1, b_2=1, guess b_1=1

i	b_i = 0	b_i = 1	Comp	Meas
3	$y = y^2 \bmod N$	$y = y^2 * M1 \bmod N$		
2	$y = y^2 \bmod N$	$y = y^2 * M1 \bmod N$		
1	$y = y^2 \bmod N$	$y = y^2 * M1 \bmod N$	X1 secs	
0	$y = y^2 \bmod N$	$y = y^2 * M1 \bmod N$		Y1 secs

i	b_i = 0	b_i = 1	Comp	Meas
3	$y = y^2 \bmod N$	$y = y^2 * M2 \bmod N$		
2	$y = y^2 \bmod N$	$y = y^2 * M2 \bmod N$		
1	$y = y^2 \bmod N$	$y = y^2 * M2 \bmod N$	X2 secs	
0	$y = y^2 \bmod N$	$y = y^2 * M2 \bmod N$		Y2 secs

Timing attacks

- If b_1 = 1, then set of { Yj - Xj | j in {1,2, ..} } has distribution with "small" variance (due to time for final step, i=0)
  - "Guess" was correct when we computed X1, X2, ...
- If b_1 = 0, then set of { Yj - Xj | j in {1,2, ..} } has distribution with "large" variance (due to time for final step, i=0, and incorrect guess for b_1)
  - "Guess" was incorrect when we computed X1, X2, ...
  - So time computation wrong (Xj computed as large, but really small, ...)
- Strategy: Force user to sign large number of messages M1, M2, ... Record timings for signing.
- Iteratively learn bits of key by using above property.