CSE 521: Design and Analysis of Algorithms
Assignment #4
April 26, 2002
Due: Friday, May 3

Reading Assignment: Kleinberg and Tardos, Chapters 5 and 6

Problems:

1. Suppose you're looking at a flow network G with source s and sink t, and you want to be able to express something like the following intuitive notion: some nodes are clearly on the "source side" of the main bottlenecks; some nodes are clearly on the "sink side" of the main bottlenecks; and some nodes are in the middle. However, G can have many minimum cuts, so we have to be careful in how we try making this idea precise.

   Here's one way to divide the nodes of G into three categories of this sort.

   - We say a node v is *upstream* if for all minimum s-t cuts (A, B), we have $v \in A$ — that is, v lies on the source side of every minimum cut.
   - We say a node v is *downstream* if for all minimum s-t cuts (A, B), we have $v \in B$ — that is, v lies on the sink side of every minimum cut.
   - We say a node v is *central* if it is neither upstream nor downstream; there is at least one minimum s-t cut (A, B) for which $v \in A$, and at least one minimum s-t cut (A', B') for which $v \in B'$.

   Give an algorithm that takes a flow network G, and classifies each of its nodes as being upstream, downstream, or central. The running time of your algorithm should be within in a constant factor of the time required to compute a *single* maximum flow.

2. Consider the following definition. We are given a set of n countries that are engaged in trade with one another. For each country i, we have the value s_i of its budget surplus; this number may be positive or negative, with a negative number indicating a deficit. For each pair of countries i, j, we have the total value e_{ij} of all exports from i to j; this number is always non-negative. We say that a subset S of the countries is *free-standing* if the sum of the budget surpluses of the countries in S, minus the total value of all exports from countries in S to countries not in S, is non-negative.

   Give a polynomial-time algorithm that takes this data for a set of n countries, and decides whether it contains a non-empty free-standing subset that is not equal to the full set.

3. Let $G = (V, E)$ be a flow network. Show that there is a sequence of at most $|E|$ augmenting paths such that augmenting along these paths in the specified order produces a maximum flow.

4. Consider an implementation of the Ford-Fulkerson max flow algorithm that at each step augments along a path that has the maximum bottleneck capacity. Give the most efficient implementation of this algorithm that you can. What is the best bound you can give on its running time?

5. Some friends of yours have grown tired of the game "Six degrees of Kevin Bacon" (after all, they ask, isn't it just breadth-first search?) and decide to invent a game with a little more punch, algorithmically speaking. Here's how it works.

   You start with a set X of n actresses and a set Y of n actors, and two players P_0 and P_1. P_0 names an actress $x_1 \in X$, P_1 names an actor y_1 who has appeared in a movie with x_1, P_0 names an actress x_2 who has appeared in a movie with y_1, and so on. Thus, P_0 and P_1 collectively generate a sequence $x_1, y_1, x_2, y_2, \ldots$ such that each actor/actress in the sequence has co-starred with the actress/actor immediately preceding. A player P_i $(i = 0, 1)$ loses when it is P_i's turn to move, and he/she cannot name a member of his/her set who hasn't been named before.

   Suppose you are given a specific pair of such sets X and Y, with complete information on who has appeared in a movie with whom. A *strategy* for P_i, in our setting, is an algorithm that takes a current sequence $x_1, y_1, x_2, y_2, \ldots$ and generates a legal next move for P_i (assuming it's P_i's turn to move). Give a polynomial-time algorithm that decides which of the two players can force a win, in a particular instance of this game.

6. **Extra Credit:** Give feedback on chapter 6 of book. Send this by email to Anna and Gideon.