CSE 521: Design and Analysis of Algorithms
Assignment #6
Due: Wednesday, February 16

**Reading Assignment:** Linear programming handouts, chapter on randomized algorithms, other randomized algorithms handouts (we'll send out mail about them.)
**Problems:**

1. Drawing graphs nicely is a problem that arises constantly in applications. Consider the problem of drawing a tree. Some characteristics that would be desirable in the drawing are:

   - All nodes on the same level in the tree should line up horizontally.
   - The vertical distance between a node and its children in the tree should not be less than some minimum value $m$.
   - All nodes should lie within a certain window on the screen.
   - The parent of a set of nodes should be centered over those nodes in the horizontal direction.
   - The height and width of the tree drawing should be small.

   How would you formulate the problem of placing the tree nodes in the drawing using linear programming? (The problem statement has purposefully been left somewhat vague. It is up to you to formalize both the problem and the solution.)

2. A multicommodity flow network supports the flow of $p$ different commodities between a set of $p$ source vertices $S = \{s_1, \ldots, s_p\}$ and $p$ sink vertices $T = \{t_1, \ldots, t_p\}$. For any edge $(u, v)$ the net flow of the $i$th commodity from $u$ to $v$ is denoted $f_i(u, v)$. For the $i$th commodity, the only source is $s_i$ and the only sink is $t_i$. There is flow conservation independently for each commodity: the net flow of each commodity out of each vertex is zero unless the vertex is the source or sink for the commodity. The sum of the net flows of all commodities on an edge $(u, v)$ must not exceed the capacity of the edge $c(u, v)$, and in this way the commodity flows interact. The value of the flow of each commodity is the net flow out of the source for that commodity. The *total flow value* is the sum of the values for all $p$ commodity flows.

   - Give a linear programming formulation for maximizing the total flow value in a given multicommodity flow network.
   - Give the dual program.

3. Use the simplex algorithm to solve the following linear program:

maximize $3x_1 + 2x_2 + 4x_3$ subject to the constraints:

$x_1 + x_2 + 2x_3 \leq 4$

$2x_1 + 3x_3 \leq 5$

$2x_1 + x_2 + 3x_3 \leq 7$

$x_1, x_2, x_3 \geq 0.$

Show the steps of the algorithm.

4. Use duality to prove the following theorem:

   Let $M$ be an $m$ by $n$ matrix. Let $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ represent probability vectors (i.e. $\sum_i p_i = \sum_j q_j = 1$ and all $p_i$ and $q_j$ nonnegative).

   Then
$$\max_{\mathbf{p}} \min_j \sum_{1 \leq i \leq m} p_i M_{ij} = \min_{\mathbf{q}} \max_i \sum_{1 \leq j \leq n} M_{ij} q_j.$$

   **Remark:** This is the von Neumann minimax theorem for 2-person zero sum games.