CSE 521: Design and Analysis of Algorithms I

Divide and Conquer

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Algorithm Design Techniques

- Divide & Conquer
  - Reduce problem to one or more sub-problems of the same type
  - Typically, each sub-problem is **at most a constant fraction** of the size of the original problem
    - e.g. Mergesort, Binary Search, Strassen’s Algorithm, Quicksort (kind of)

Fast exponentiation

- Power(**a**,**n**)
  - **Input:** integer **n** and number **a**
  - **Output:** a^n
- Obvious algorithm
  - **n-1** multiplications
- Observation:
  - if **n** is even, **n=2m**, then $a^n=a^m \bullet a^m$

Divide & Conquer Algorithm

- Power(**a**,**n**)

```
if $n=0$ then return($1$)
else if $n=1$ then return($a$)
else
    $x \leftarrow$ Power($a$,$\lfloor n/2 \rfloor$)
    if $n$ is even then
        return($x \cdot x$)
    else
        return($a \cdot x \cdot x$)
```

Analysis

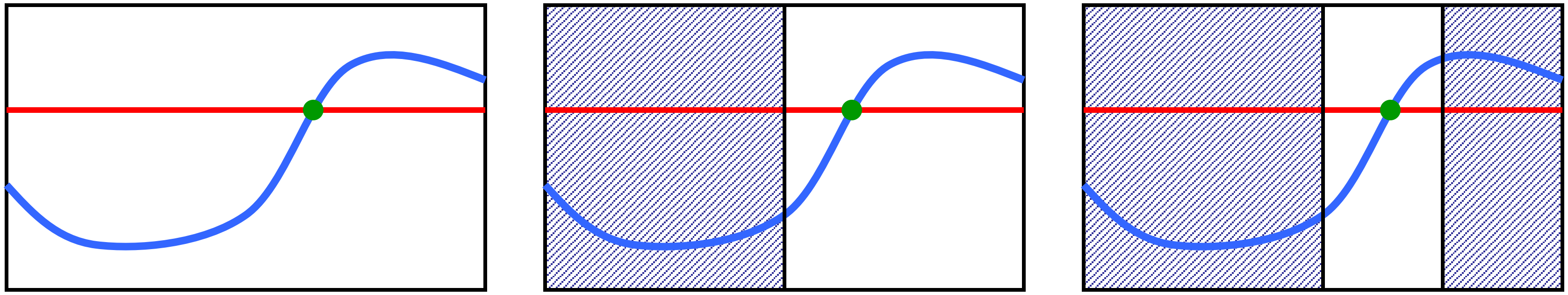
- Worst-case recurrence
  - $T(n)=T(\lfloor n/2 \rfloor)+2$ for $n \geq 1$
  - $T(1)=0$
- Time
  - $T(n)=T(\lfloor n/2 \rfloor)+2 \leq T(\lfloor n/4 \rfloor)+2+2 \leq \ldots$
    $\leq T(1)+\underbrace{2+\ldots+2}_{\log_2 n \text{ copies}} = 2 \log_2 n$
- More precise analysis:
  - $T(n)= \lceil \log_2 n \rceil$ + # of 1's in n's binary representation

A Practical Application- RSA

- Instead of **a^n** want **a^n mod N**
  - a^{i+j} mod N = ((a^i mod N)•(a^j mod N)) mod N
  - same algorithm applies with each **x•y** replaced by
    - ((x mod N)•(y mod N)) mod N

- In RSA cryptosystem
  - need **a^n mod N** where **a, n, N** each typically have **1024** bits
  - Power: at most **2048** multiplies of **1024** bit numbers
    - relatively easy for modern machines
  - Naive algorithm: **2^{1024}** multiplies

Binary search for roots (bisection method)



- Given:
  - continuous function **f** and two points **a**<**b** with **f(a)** ≤ **0** and **f(b)** > **0**
- Find:
  - approximation to **c** s.t. **f(c)=0** and **a**<**c**<**b**

Bisection method

```
Bisection($\mathbf{a}$,$\mathbf{b}$, $\varepsilon$)
    if ($\mathbf{a}$-$\mathbf{b}$) $< \varepsilon$ then
        return($\mathbf{a}$)
    else
        $\mathbf{c} \leftarrow (\mathbf{a}+\mathbf{b})/\mathbf{2}$
        if $\mathbf{f}(\mathbf{c}) \leq \mathbf{0}$ then
            return(Bisection($\mathbf{c}$,$\mathbf{b}$,$\varepsilon$))
        else
            return(Bisection($\mathbf{a}$,$\mathbf{c}$,$\varepsilon$))
```

Time Analysis

- At each step we halved the size of the interval
- It started at size **b-a**
- It ended at size ε

- # of calls to f is $\log_2((\mathbf{b-a})/\varepsilon)$

Old favorites

- Binary search
  - One subproblem of half size plus one comparison
  - Recurrence $T(n) = T(\lceil n/2 \rceil)+1$ for $n \geq 2$

    $T(1) = 0$

  So $T(n)$ is $\lceil \log_2 n \rceil+1$
- Mergesort
  - Two subproblems of half size plus merge cost of $n-1$ comparisons
  - Recurrence $T(n) \leq 2T(\lceil n/2 \rceil)+n-1$ for $n \geq 2$

    $T(1) = 0$

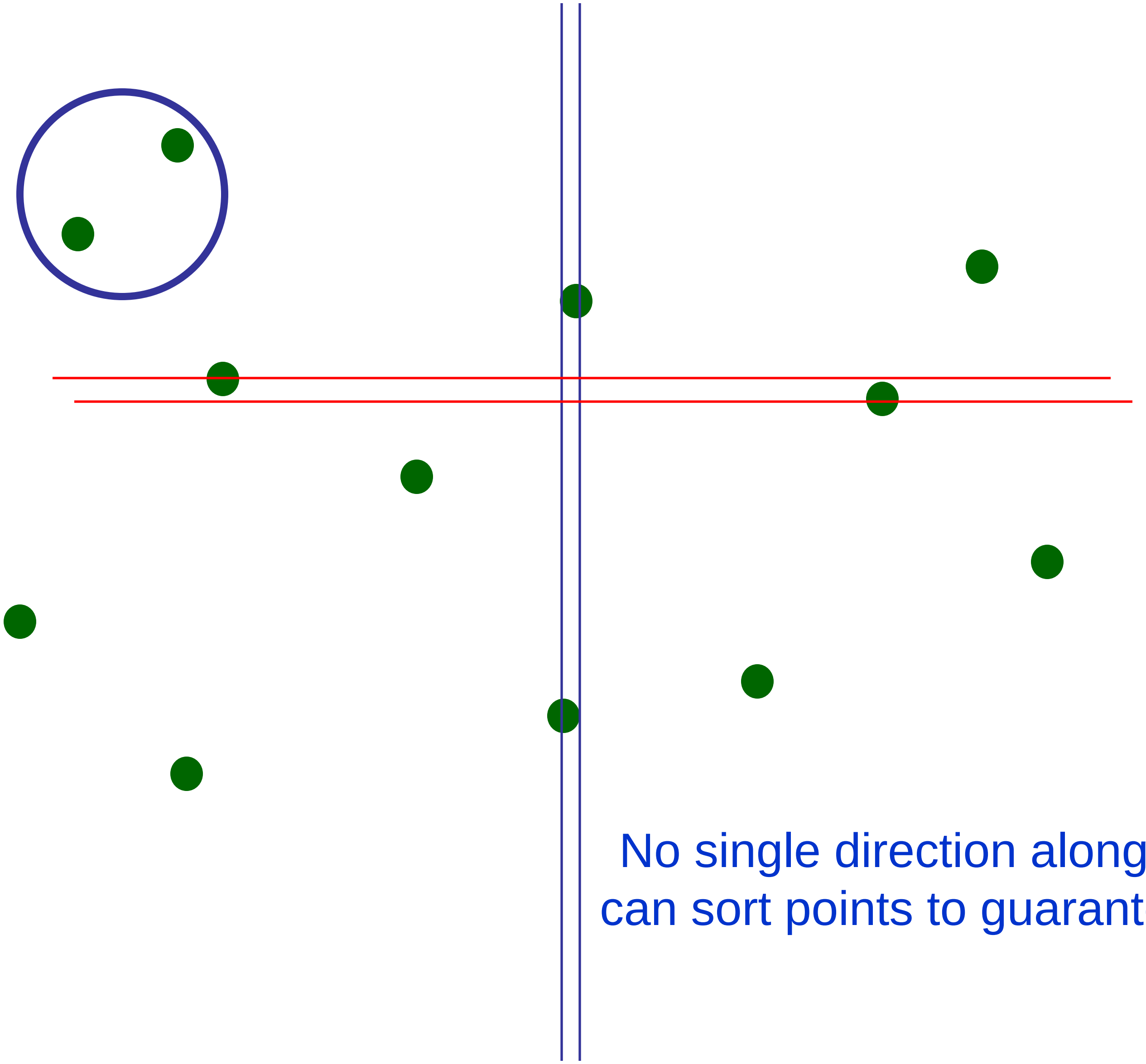
    Roughly n comparisons at each of $\log_2 n$ levels of recursion

  So $T(n)$ is roughly $2n \log_2 n$

Euclidean Closest Pair

- Given a set **P** of n points $\mathbf{p_1,\ldots,p_n}$ with real-valued coordinates
- Find the pair of points $\mathbf{p_i,p_j} \in \mathbf{P}$ such that the Euclidean distance $\mathbf{d(p_i,p_j)}$ is minimized
- $\Theta(\mathbf{n^2})$ possible pairs
- In one dimension there is an easy $\mathbf{O(n \log n)}$ algorithm
  - Sort the points
  - Compare consecutive elements in the sorted list
- What about points in the plane?

Closest Pair in the Plane

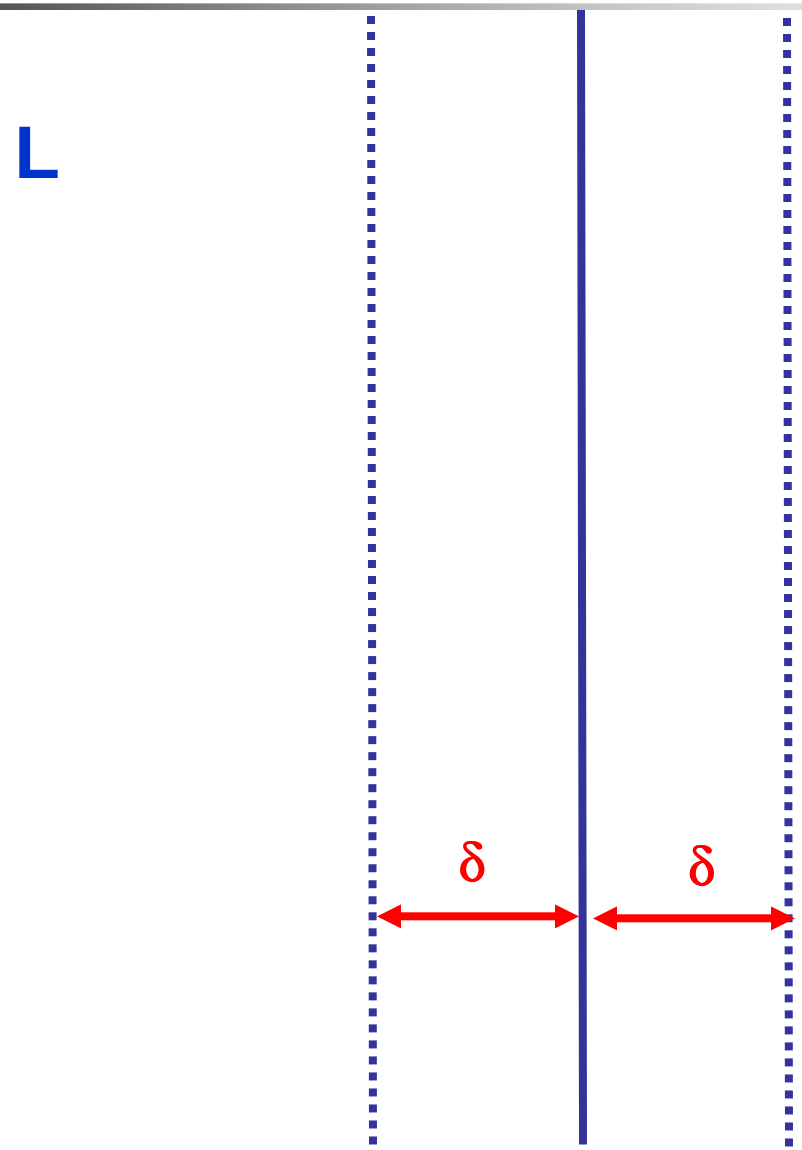


No single direction along which one can sort points to guarantee success!

Closest Pair In the Plane: Divide and Conquer

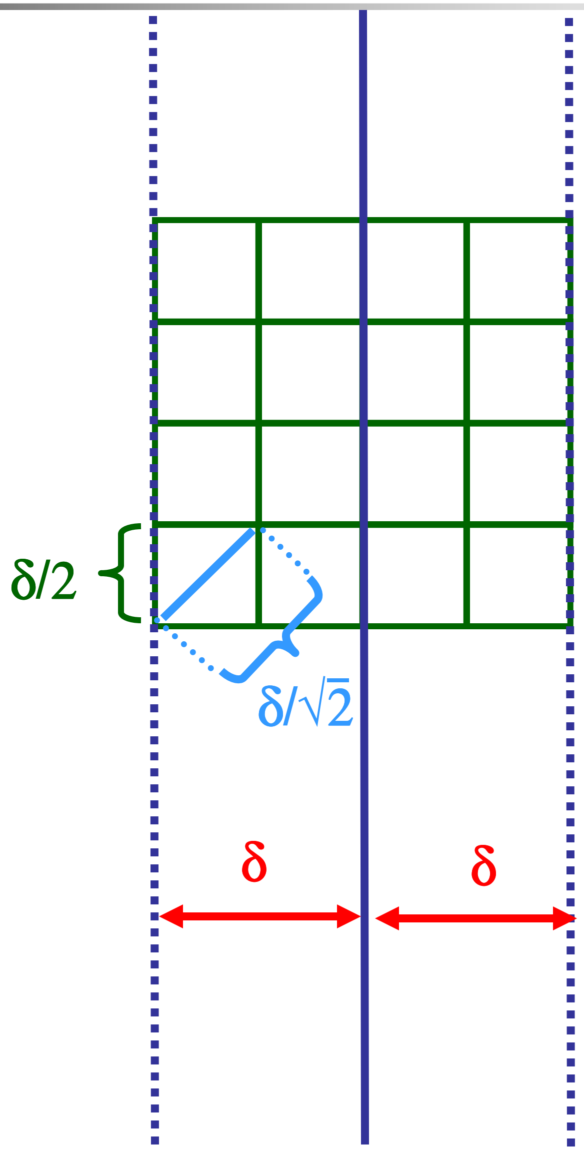
- Sort the points by their **x** coordinates
- Split the points into two sets of **n**/2 points **L** and **R** by **x** coordinate
- Recursively compute
  - closest pair of points in **L**, (p_L,q_L)
  - closest pair of points in **R**, (p_R,q_R)
- Let $\delta=\min\{d(p_L,q_L),d(p_R,q_R)\}$ and let (p,q) be the pair of points that has distance δ
- This may not be enough!
  - Closest pair of points may involve one point from **L** and the other from **R**!

A clever geometric idea



Any pair of points **p**∈**L** and **q**∈**R** with **d(p,q)**<δ must lie in band

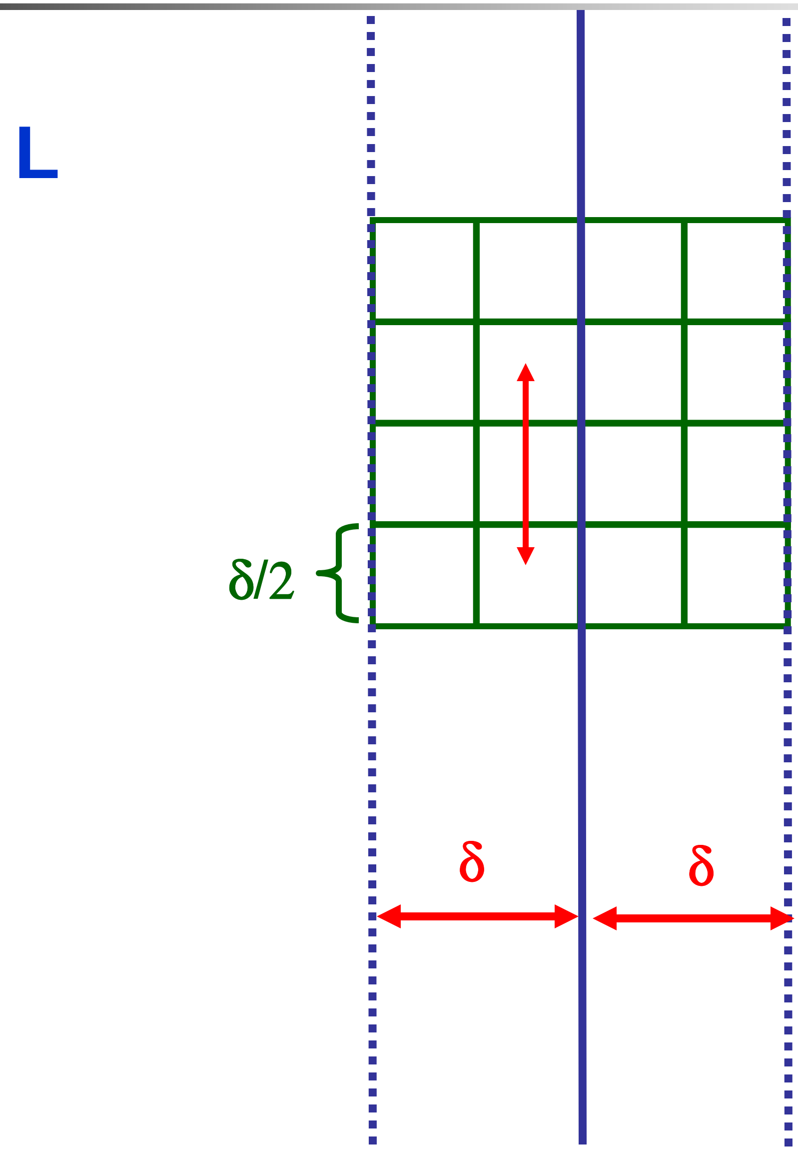
A clever geometric idea



Any pair of points **p**$\in$**L** and **q**$\in$**R** with **d(p,q)**$<\delta$ must lie in band

No two points can be in the same green box

A clever geometric idea



Any pair of points $p \in L$ and $q \in R$ with $d(p,q) < \delta$ must lie in band

No two points can be in the same green box

Only need to check pairs of points up to 2 rows apart -
a constant # of other points!

Closest Pair Recombining

- Sort points by **y** coordinate ahead of time
- On recombination only compare each point in **L∪R** to the **12** points above it in the **y** sorted order
- If any of those distances is better than **δ** replace **(p,q)** by the best of those pairs
- **O(n log n)** for **x** and **y** sorting at start
- Two recursive calls on problems on half size
- **O(n)** recombination
- Total **O(n log n)**

Sometimes two sub-problems aren't enough

- More general divide and conquer
  - You've broken the problem into **a** different sub-problems
  - Each has size at most **n/b**
  - The cost of the break-up and recombining the sub-problem solutions is $\mathbf{O(n^k)}$

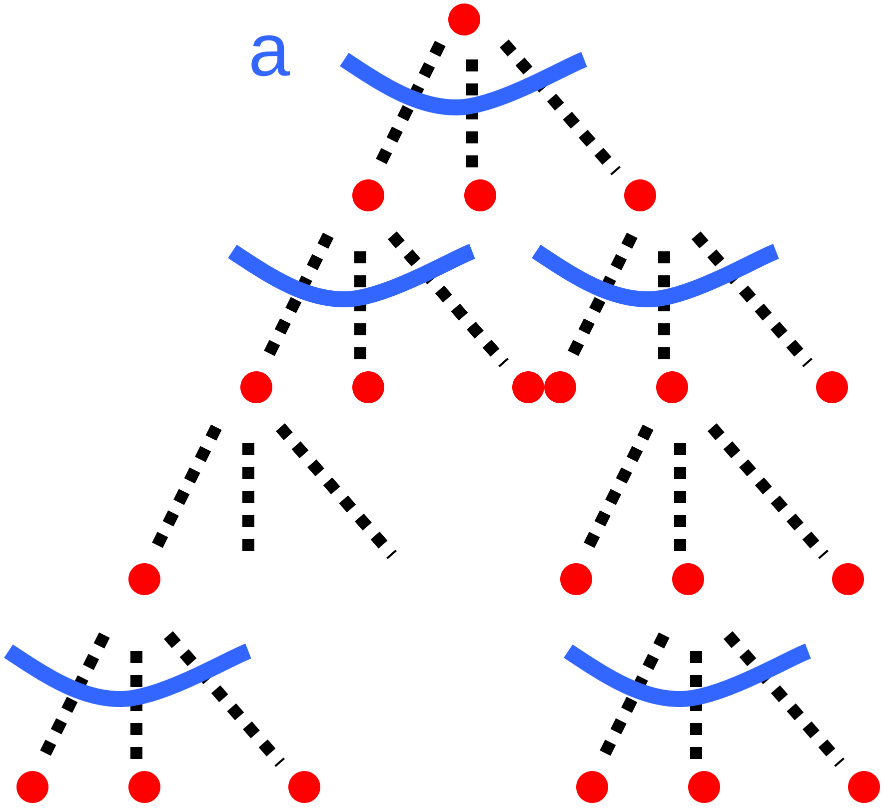
- Recurrence
  - $\mathbf{T(n) \leq a \cdot T(n/b) + c \cdot n^k}$

Master Divide and Conquer Recurrence

- If $T(n) \leq a \cdot T(n/b) + c \cdot n^k$ for $n > b$ then
  - if $a > b^k$ then $T(n)$ is $\Theta(n^{\log_b a})$
  - if $a < b^k$ then $T(n)$ is $\Theta(n^k)$
  - if $a = b^k$ then $T(n)$ is $\Theta(n^k \log n)$
- Works even if it is $\lceil n/b \rceil$ instead of n/b.

Proving Master recurrence

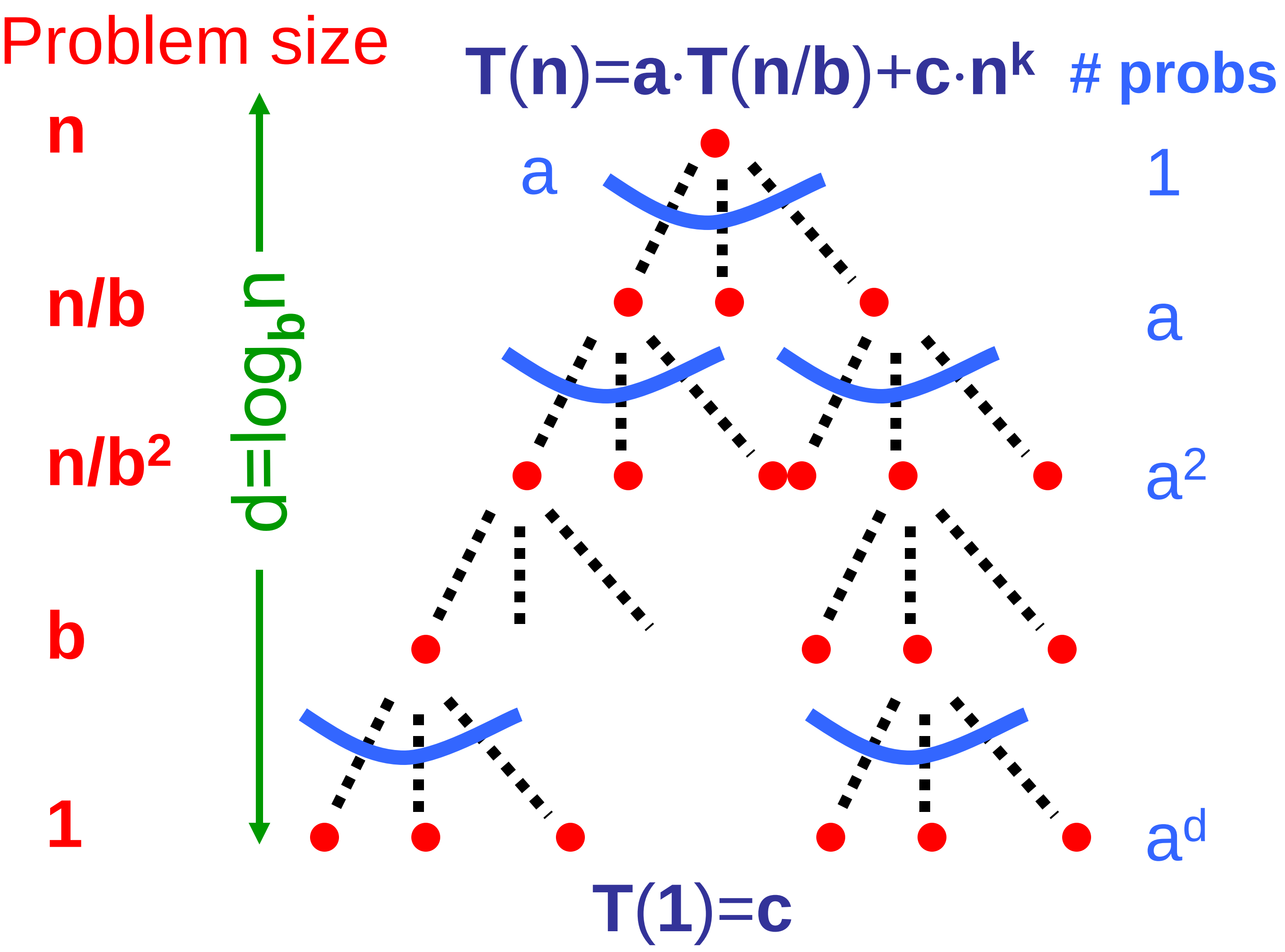
$T(n)=aT(n/b)+cn^k$



Problem size	# probs
n	1
n/b	a
n/b^2	a^2
b	
1	a^d

$T(1)=c$

Proving Master recurrence



Problem size

$T(n)=a{\cdot}T(n/b)+c{\cdot}n^k$

 # probs

n a 1

n/b a

n/b^2 a^2

b

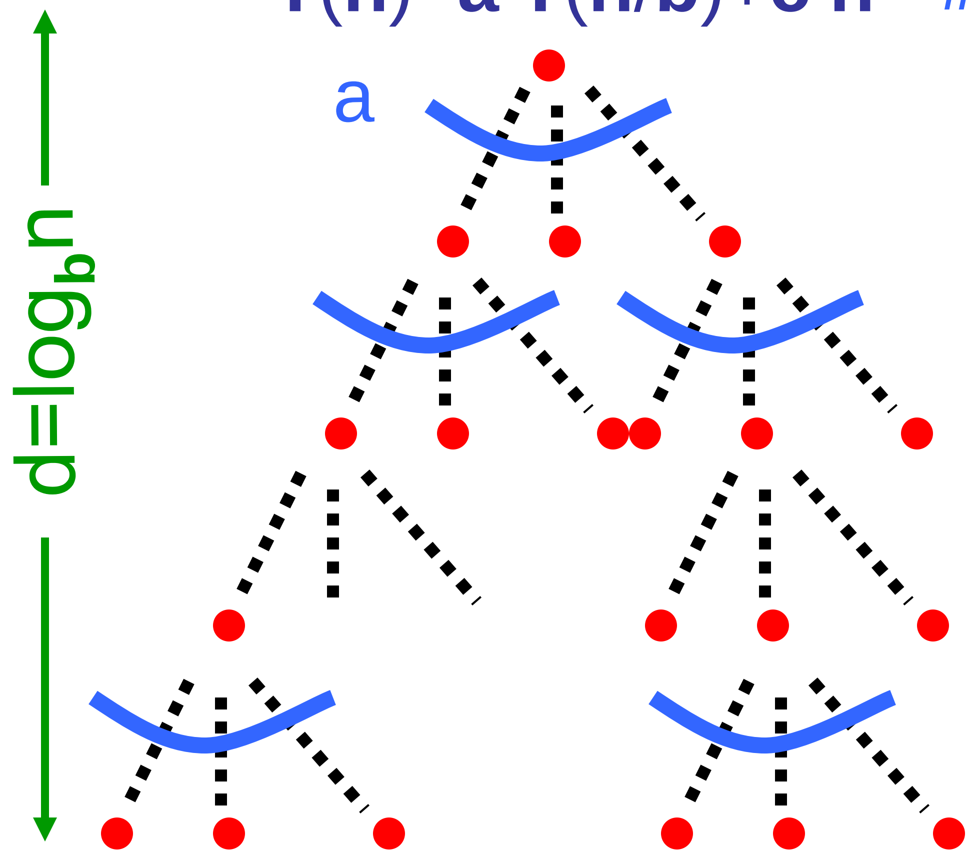
1 a^d

$d=\log_b n$

T(1)=c

Proving Master recurrence

$T(n)=a \cdot T(n/b)+c \cdot n^k$



Problem size	# probs	cost
n	1	cn^k
n/b	a	$c \cdot a \cdot n^k/b^k$
n/b^2	a^2	$c \cdot a^2 \cdot n^k/b^{2k}$ $=c \cdot n^k(a/b^k)^2$
b		
1	a^d	$c \cdot n^k(a/b^k)^d$ $=c \cdot a^d$

$T(1)=c$

Geometric Series

- $S = t + tr + tr^2 + \ldots + tr^{n-1}$
- $r \cdot S = tr + tr^2 + \ldots + tr^{n-1} + tr^n$
- $(r-1)S = tr^n - t$
- so $S = t(r^n - 1)/(r-1)$ if $r \neq 1$.

- Simple rule
  - If $r \neq 1$ then S is a constant times largest term in series

Total Cost

- Geometric series
  - ratio **a/b^k**
  - **$d+1=\log_b n +1$** terms
  - first term **cn^k**, last term **ca^d**
- If **$a/b^k=1$**
  - all terms are equal **$T(n)$** is $\Theta(n^k \log n)$
- If **$a/b^k<1$**
  - first term is largest **$T(n)$** is $\Theta(n^k)$
- If **$a/b^k>1$**
  - last term is largest **$T(n)$** is $\Theta(a^d)=\Theta(a^{\log_b n}) =\Theta(n^{\log_b a})$

  (To see this take $\log_b$ of both sides)

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

- n^3 multiplications, n^3-n^2 additions

Multiplying Matrices

```
for $i=1$ to $n$
    for $j=1$ to $n$
        $C[i,j] \leftarrow 0$
        for $k=1$ to $n$
            $C[i,j]=C[i,j]+A[i,k]\cdot B[k,j]$
        endfor
    endfor
endfor
```

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

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Multiplying Matrices

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Multiplying Matrices

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \bullet \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right]$$

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \bullet \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ \hline a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{array}\right]$$

$$\left[\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ \hline A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right]$$

Simple Divide and Conquer

$$\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)\left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

$$= \left(\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ \hline A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right)$$

- $T(n)=8T(n/2)+4(n/2)^2=8T(n/2)+n^2$
  - $8>2^2$ so $T(n)$ is

$$\Theta(n^{\log_b a}) = \Theta(n^{\log_2 8}) = \Theta(n^3)$$

Strassen's Divide and Conquer Algorithm

- Strassen's algorithm
  - Multiply 2x2 matrices using 7 instead of 8 multiplications (and lots more than 4 additions)

  - $T(n)=7\ T(n/2)+cn^2$
    - $7>2^2$ so $T(n)$ is $\Theta(n^{\log_2 7})$ which is $O(n^{2.81\ldots})$

  - Fastest algorithms theoretically use $O(n^{2.376})$ time
    - not practical but Strassen's is practical **provided calculations are exact** and we stop recursion when matrix has size about 32 and use ordinary multiplication for subproblems

The algorithm

$P_1 \leftarrow A_{12}(B_{11}+B_{21})$; $P_2 \leftarrow A_{21}(B_{12}+B_{22})$

$P_3 \leftarrow (A_{11} - A_{12})B_{11}$; $P_4 \leftarrow (A_{22} - A_{21})B_{22}$

$P_5 \leftarrow (A_{22} - A_{12})(B_{21} - B_{22})$

$P_6 \leftarrow (A_{11} - A_{21})(B_{12} - B_{11})$

$P_7 \leftarrow (A_{21} - A_{12})(B_{11}+B_{22})$

$C_{11} \leftarrow P_1+P_3$; $C_{12} \leftarrow P_2+P_3+P_6 - P_7$

$C_{21} \leftarrow P_1+P_4+P_5+P_7$; $C_{22} \leftarrow P_2+P_4$

Another Divide &Conquer Example: Multiplying Faster

- If you analyze our usual elementary school algorithm for multiplying numbers
  - $\Theta(n^2)$ time
  - On real machines each “digit” is, e.g., **32** bits long but still get $\Theta(n^2)$ running time with this algorithm when run on **n**-bit multiplication
- We can do better!
  - We’ll describe the basic ideas by multiplying polynomials rather than integers
  - Advantage is we don’t get confused by worrying about carries

Polynomial Multiplication

- **Given:**
  - Degree **n-1** polynomials **P** and **Q**
    - $P = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$
    - $Q = b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-2}x^{n-2} + b_{n-1}x^{n-1}$
- **Compute:**
  - Degree **2n-2** Polynomial **P Q**
  - $P\,Q = a_0b_0 + (a_0b_1+a_1b_0)\,x + (a_0b_2+a_1b_1+a_2b_0)\,x^2 + \ldots + (a_{n-2}b_{n-1}+a_{n-1}b_{n-2})\,x^{2n-3} + a_{n-1}b_{n-1}\,x^{2n-2}$
- **Obvious Algorithm:**
  - Compute all a_ib_j and collect terms
  - $\Theta(n^2)$ time

Naive Divide and Conquer

- Assume **n=2k**
  - **P** = (a_0 + a_1 x + a_2 x^2 + … + a_{k-2} x^{k-2} + a_{k-1} x^{k-1}) + (a_k + a_{k+1} x + … + $a_{n-2}x^{k-2}$ + $a_{n-1}x^{k-1}$) x^k

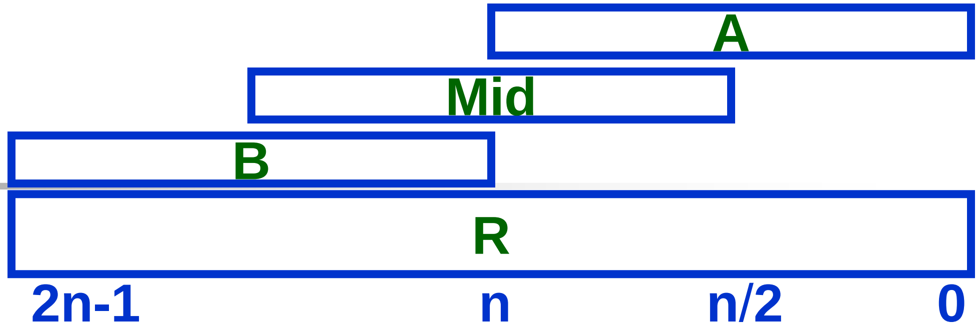
    = P_0 + P_1 x^k where P_0 and P_1 are degree **k-1** polynomials
  - Similarly **Q** = Q_0 + Q_1 x^k
- **P Q** = $(P_0+P_1x^k)(Q_0+Q_1x^k)$

  = $P_0Q_0 + (P_1Q_0+P_0Q_1)x^k + P_1Q_1x^{2k}$
- **4** sub-problems of size **k=n/2** plus linear combining
  - $T(n)=4{\cdot}T(n/2)+cn$ Solution $T(n) = \Theta(n^2)$

Karatsuba's Algorithm

- A better way to compute the terms
  - Compute
    - $A \leftarrow P_0Q_0$
    - $B \leftarrow P_1Q_1$
    - $C \leftarrow (P_0+P_1)(Q_0+Q_1) = P_0Q_0+P_1Q_0+P_0Q_1+P_1Q_1$
  - Then
    - $P_0Q_1+P_1Q_0 = C-A-B$
    - So $PQ=A+(C-A-B)x^k+Bx^{2k}$
  - 3 sub-problems of size n/2 plus O(n) work
    - $T(n) = 3\ T(n/2) + cn$
    - $T(n) = O(n^\alpha)$ where $\alpha = \log_2 3 = 1.59...$

Karatsuba: Details

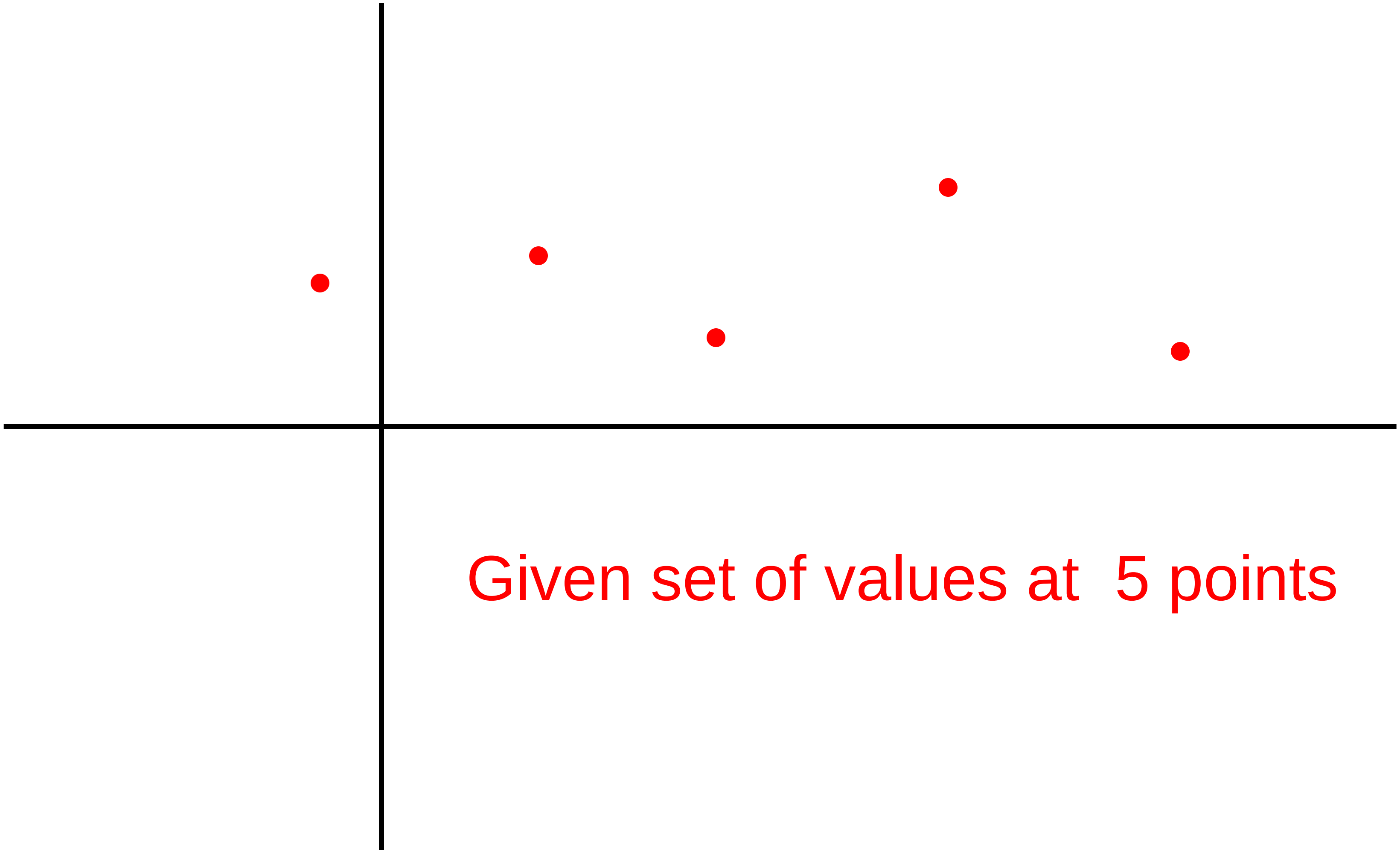


```
PolyMul(P, Q):
    // P, Q are length n =2k vectors, with P[i], Q[i] being
    // the coefficient of $x^i$ in polynomials P, Q respectively.
    // Let Pzero be elements 0..k-1 of P; Pone be elements k..n-1
    // Qzero, Qone : similar
    If n=1 then Return(P[0]*Q[0]) else
      A ← PolyMul(Pzero, Qzero);      // result is a (2k-1)-vector
      B ← PolyMul(Pone, Qone);        // ditto
      Psum ← Pzero + Pone;            // add corresponding elements
      Qsum ← Qzero + Qone;            // ditto
      C ← polyMul(Psum, Qsum);        // another (2k-1)-vector
      Mid ← C – A – B;                // subtract correspond elements
      R ← A + Shift(Mid, n/2) +Shift(B,n)   // a (2n-1)-vector
      Return( R);
```

Multiplication

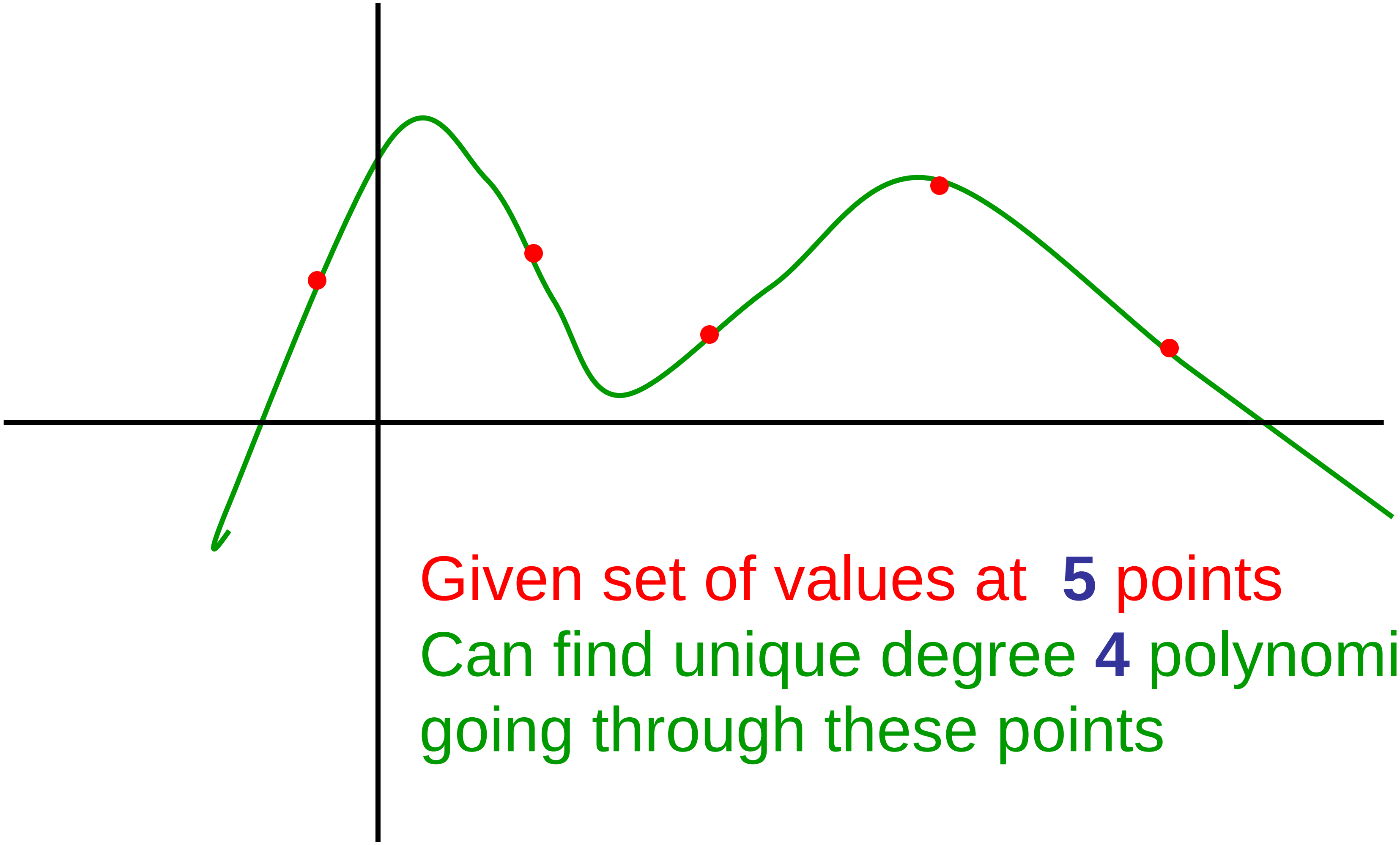
- Polynomials
  - Naïve: $\Theta(n^2)$
  - Karatsuba: $\Theta(n^{1.59\ldots})$
  - Best known: $\Theta(n \log n)$
    - "Fast Fourier Transform“
    - FFT widely used for signal processing, especially as DCT (Discrete Cosine Transform)
- Integers
  - Similar, but some ugly details re: carries, etc. due to Schonhage-Strassen in 1971 gives $\Theta(n \log n \operatorname{loglog} n)$,
  - Improvement in 2007 due to Furer gives $\Theta(n \log n\ 2^{\log^* n})$
  - Used in practice in symbolic manipulation systems like Maple

Hints towards FFT: Interpolation



Given set of values at 5 points

Hints towards FFT: Interpolation



Given set of values at **5** points

Can find unique degree **4** polynomial going through these points

Multiplying Polynomials by Evaluation & Interpolation

- Any degree **$n-1$** polynomial **$R(y)$** is determined by **$R(y_0)$, … $R(y_{n-1})$** for any **n** distinct **$y_0,\dots,y_{n-1}$**

- To compute **PQ** (assume degree at most **$n-1$**)
  - Evaluate **$P(y_0),\dots, P(y_{n-1})$**
  - Evaluate **$Q(y_0),\dots,Q(y_{n-1})$**
  - Multiply values **$P(y_i)Q(y_i)$** for **$i=0,\dots,n-1$**
  - Interpolate to recover **PQ**

Interpolation

- Given values of degree **n-1** polynomial **R** at **n** distinct points $y_1,\dots,y_n$
  - $R(y_1),\dots,R(y_n)$
- Compute coefficients $c_0,\dots,c_{n-1}$ such that
  - $R(x)=c_0+c_1x+c_2x^2+\dots+c_{n-1}x^{n-1}$
- System of linear equations in $c_0,\dots,c_{n-1}$

$c_0 + c_1 y_1 + c_2 y_1^2 + \dots + c_{n-1} y_1^{n-1} = R(y_1)$

known

$c_0 + c_1 y_2 + c_2 y_2^2 + \dots + c_{n-1} y_2^{n-1} = R(y_2)$

…

unknown

$c_0 + c_1 y_n + c_2 y_n^2 + \dots + c_{n-1} y_n^{n-1} = R(y_n)$

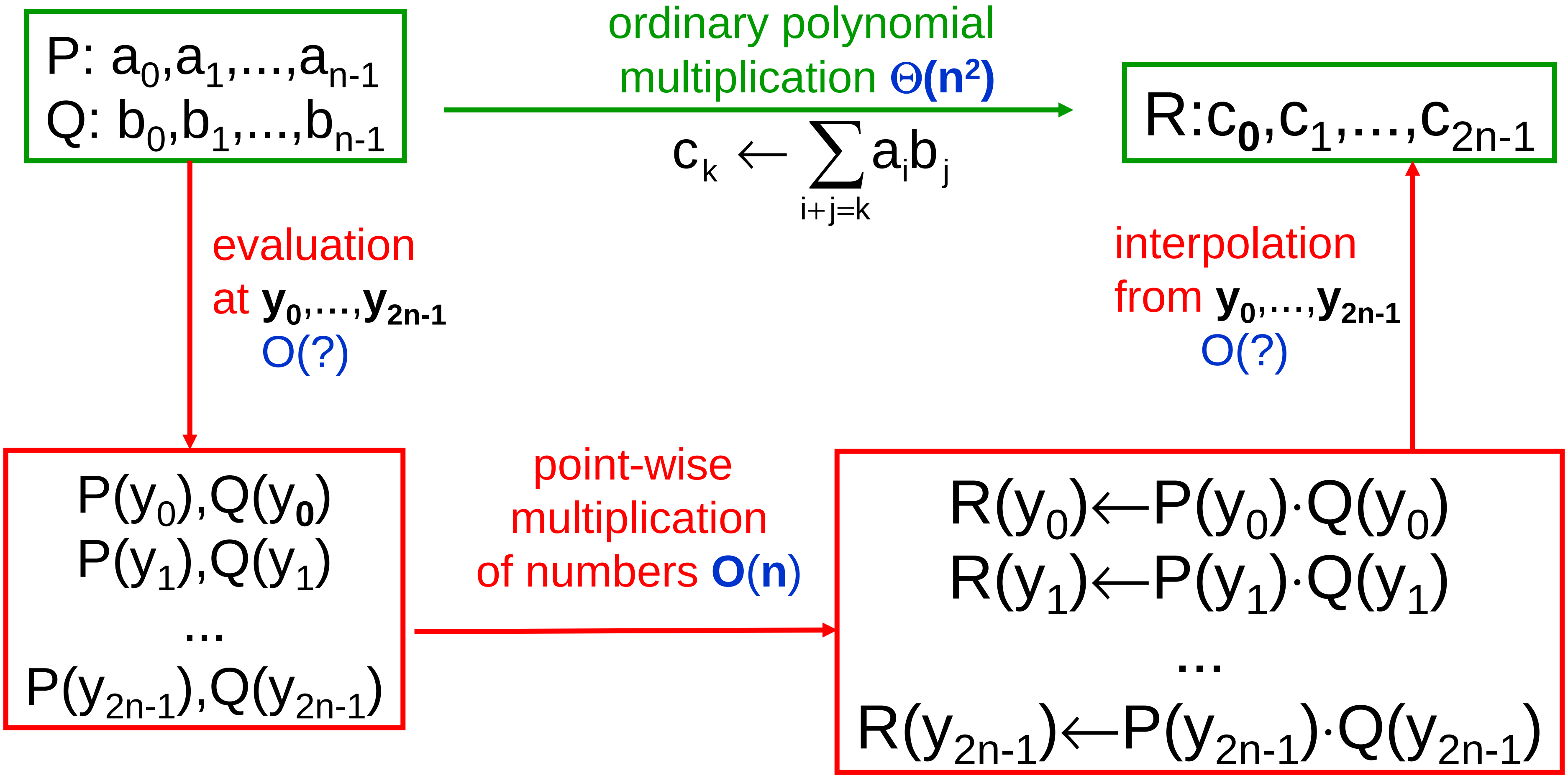
Interpolation:
n equations in n unknowns

- Matrix form of the linear system

$$\begin{pmatrix} 1 & y_1 & y_1^2 & \dots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \dots & y_2^{n-1} \\ & \dots & & & \\ & \dots & & & \\ 1 & y_n & y_n^2 & \dots & y_n^{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} R(y_1) \\ R(y_2) \\ . \\ . \\ R(y_n) \end{pmatrix}$$

- Fact: Determinant of the matrix is $\prod_{i<j} (y_i - y_j)$ which is not **0** since points are distinct
  - System has a unique solution $c_0, \dots, c_{n-1}$

Hints towards FFT: Evaluation & Interpolation



P: $a_0,a_1,\dots,a_{n-1}$
Q: $b_0,b_1,\dots,b_{n-1}$

ordinary polynomial multiplication **$\Theta(n^2)$**

$$c_k \leftarrow \sum_{i+j=k} a_i b_j$$

R:$c_{\mathbf{0}},c_1,\dots,c_{2n-1}$

evaluation
at $\mathbf{y_0,\dots,y_{2n-1}}$
O(?)

interpolation
from $\mathbf{y_0,\dots,y_{2n-1}}$
O(?)

$P(y_0),Q(y_{\mathbf{0}})$
$P(y_1),Q(y_1)$
...
$P(y_{2n-1}),Q(y_{2n-1})$

point-wise
multiplication
of numbers **O**(**n**)

$R(y_0)\leftarrow P(y_0)\cdot Q(y_0)$
$R(y_1)\leftarrow P(y_1)\cdot Q(y_1)$
...
$R(y_{2n-1})\leftarrow P(y_{2n-1})\cdot Q(y_{2n-1})$

Karatsuba's algorithm and evaluation and interpolation

- Strassen gave a way of doing **2**x**2** matrix multiplies with fewer multiplications
- Karatsuba's algorithm can be thought of as a way of multiplying degree **1** polynomials (which have **2** coefficients) using fewer multiplications
  - $PQ=(P_0+P_1z)(Q_0+Q_1z)$
    $= P_0Q_0 + (P_1Q_0+P_0Q_1)z + P_1Q_1z^2$
  - Alternative Karatsuba: evaluate at **0,1,-1** (Could also use other points)
    - $A = P(0)Q(0)= P_0Q_0$
    - $C = P(1)Q(1)=(P_0+P_1)(Q_0+Q_1)$
    - $D = P(-1)Q(-1)=(P_0 -P_1)(Q_0 -Q_1)$
  - Interpolating, $P_1Q_0+P_0Q_1=(C-D)/2$ and $P_1Q_1=(C+D)/2-A$

Evaluation at Special Points

- Evaluation of polynomial at **1** point takes **O(n)** time
  - So **2n** points (naively) takes **$O(n^2)$**—no savings
  - But the algorithm works no matter what the points are…
- So…choose points that are related to each other so that evaluation problems can share subproblems

The key idea:
Evaluate at related points

- $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots+a_{n-1}x^{n-1}$

  $= a_0 +a_2x^2 +a_4x^4 +\ldots+ a_{n-2}x^{n-2}$

  $+ a_1x+a_3x^3 +a_5x^5 +\ldots+a_{n-1}x^{n-1}$

  $= P_{even}(x^2) + x\ P_{odd}(x^2)$

- $P(-x)=a_0 -a_1x+a_2x^2 -a_3x^3+a_4x^4-\ldots\ -a_{n-1}x^{n-1}$

  $= a_0 +a_2x^2 +a_4x^4 +\ldots+ a_{n-2}x^{n-2}$

  $- (a_1x+a_3x^3 +a_5x^5 +\ldots+a_{n-1}x^{n-1})$

  $= P_{even}(x^2) - x\ P_{odd}(x^2)$

where $P_{even}(x) = a_0 +a_2x +a_4x^2 +\ldots+ a_{n-2}x^{n/2-1}$

and $P_{odd}(x) = a_1+a_3x +a_5x^2 +\ldots+a_{n-1}x^{n/2-1}$

The key idea: Evaluate at related points

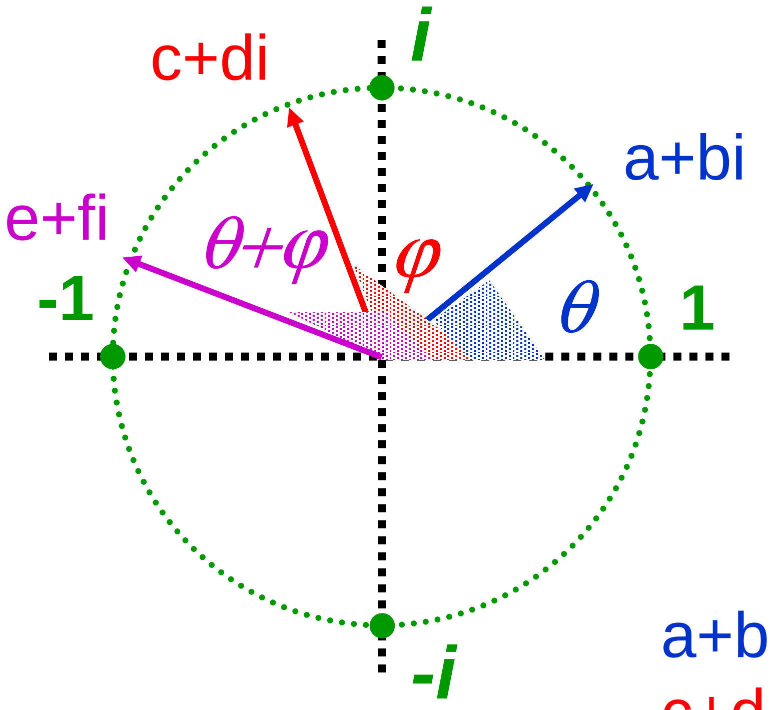
- So… if we have half the points as negatives of the other half
  - i.e., $y_{n/2} = -y_0$, $y_{n/2+1} = -y_1$,…,$y_{n-1} = -y_{n/2-1}$

  then we can reduce the size **n** problem of evaluating degree **n-1** polynomial **P** at **n** points to evaluating **2** degree **n/2-1** polynomials P_{even} and P_{odd} at **n/2** points y_0^2,…$y_{n/2-1}^2$ and recombine answers with **O(1)** extra work per point

- But to use this idea recursively we need half of y_0^2,…$y_{n/2-1}^2$ to be negatives of the other half
  - If $y_{n/4}^2 = -y_0^2$, say, then $(y_{n/4}/y_0)^2 = -1$
  - Motivates use of complex numbers as evaluation points

Complex Numbers

$i^2 = -1$



To multiply complex numbers:
1. add angles
2. multiply lengths
(all length 1 here)

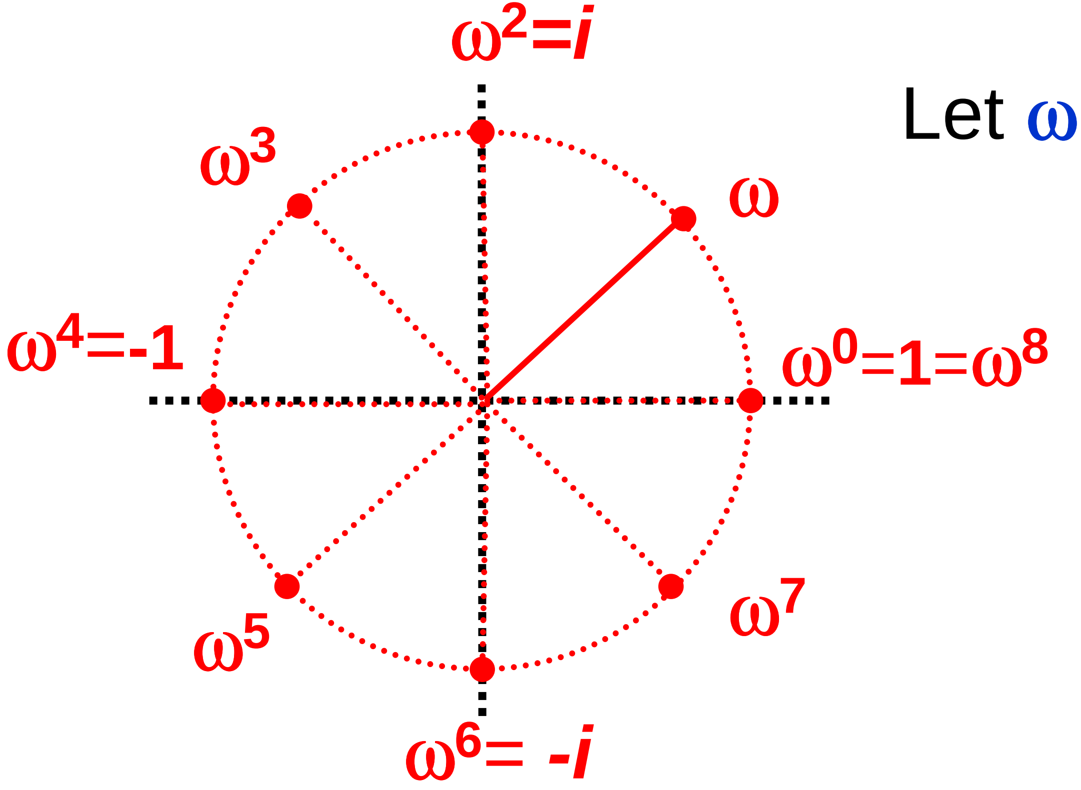
e+fi = (a+bi)(c+di)

a+bi =cos θ +i sin θ = $e^{i\theta}$
c+di =cos φ +i sin φ = $e^{i\varphi}$
e+fi =cos (θ+φ) +i sin (θ+φ) = $e^{i(\theta+\varphi)}$

$e^{2\pi i} = 1$
$e^{\pi i} = -1$

Primitive n^th^ root of 1 $\omega=\omega_n= e^{i\,2\pi/n}$

/n



Let $\omega = \omega_n = e^{i\,2\pi\,/n}$

$= \cos(2\pi/n) + i \sin(2\pi/n)$

$i^2 = -1$

$E^{2\pi\, i} = 1$

Facts about $\omega = e^{2\pi i/n}$ for even n

- $\omega = e^{2\pi i/n}$, $i = \sqrt{-1}$
- $\omega^n = 1$
- $\omega^{n/2} = -1$
- $\omega^{n/2+k} = -\omega^k$ for all values of k
- $\omega^2 = e^{2\pi i/m}$ where $m=n/2$
- $\omega^k = \cos(2k\pi/n) + i\sin(2k\pi/n)$ so can compute with powers of ω
- ω^k is a root of $x^n-1 = (x-1)(x^{n-1}+x^{n-2}+\ldots+1) = 0$ but for $k \neq 0$, $\omega^k \neq 1$ so $\omega^{k(n-1)}+\omega^{k(n-2)}+\ldots+1=0$

The recursive idea for n a power of 2

- Goal:
  - Evaluate **P** at $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$
- Now
  - P_{even} and P_{odd} have degree $n/2-1$ where
  - $P(\omega^k) = P_{even}(\omega^{2k}) + \omega^k P_{odd}(\omega^{2k})$
  - $P(-\omega^k) = P_{even}(\omega^{2k}) - \omega^k P_{odd}(\omega^{2k})$
- Recursive Algorithm
  - Evaluate P_{even} at $1, \omega^2, \omega^4, \ldots, \omega^{n-2}$
  - Evaluate P_{odd} at $1, \omega^2, \omega^4, \ldots, \omega^{n-2}$
  - Combine to compute **P** at $1, \omega, \omega^2, \ldots, \omega^{n/2-1}$
  - Combine to compute **P** at $-1, -\omega, -\omega^2, \ldots, -\omega^{n/2-1}$ (i.e. at $\omega^{n/2}, \omega^{n/2+1}, \omega^{n/2+2}, \ldots, \omega^{n-1}$)

ω^2 is $e^{2\pi i/m}$ where $m=n/2$ so problems are of same type but smaller size

Analysis and more

- Run-time
  - $\mathbf{T(n)=2 \cdot T(n/2)+cn}$ so $\mathbf{T(n)=O(n \log n)}$
- So much for evaluation ... what about interpolation?
  - Given
    - $\mathbf{R(1), R(\omega), R(\omega^2),\ldots, R(\omega^{n-1})}$
  - Compute
    - $\mathbf{c_0, c_1,\ldots,c_{n-1}}$ s.t. $\mathbf{R(x)=c_0+c_1x+\ldots+c_{n-1}x^{n-1}}$

Interpolation:
n equations in n unknowns

- Matrix form of the linear system

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ & \dots & & & \\ & \dots & & & \\ 1 & \omega^{n-1} & \omega^{2n-2} & \dots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} R(1) \\ R(\omega) \\ . \\ . \\ R(\omega^{n-1}) \end{pmatrix}$$

- Let **M** be the interpolation matrix for these points
  - That is: $\mathbf{M}_{ij} = \omega^{ij}$

The inverse of M

- Define matrix **N** by $\mathbf{N}_{ij} = \omega^{-ij}$.
- Then $(\mathbf{MN})_{ij} = \Sigma_{k=0..n-1}\, \omega^{ik}\, \omega^{-kj} = \Sigma_{k=0..n-1}\, \omega^{k(i-j)}$
- If $i=j$ then this is $\Sigma_{k=0..n-1}\, 1 = n$
- If $i \neq j$ then this is

$$1+\omega^{i-j}+\omega^{2(i-j)}+\ldots+\omega^{(n-1)(i-j)}=0$$

- So **MN** is **n** times the identity matrix; that is $\mathbf{M}^{-1} = \mathbf{N}/n$

Interpolation using FFT

- So… **$C = M^{-1} R = N R/n$;** that is

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \dots & \omega^{-(n-1)} \\ & \dots & & & \\ & \dots & & & \\ 1 & \omega^{-(n-1)} & \omega^{-(2n-2)} & \dots & \omega^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} R(1)/n \\ R(\omega)/n \\ . \\ . \\ R(\omega^{n-1})/n \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ c_{n-1} \end{pmatrix}$$

- But matrix **N** is just the matrix for the evaluation at points **$1, \omega^{-1}, \omega^{-2}, \dots, \omega^{-(n-1)}$** **!!!**
- So…apply the same FFT recursion for the interpolation phase which is also **$O(n \log n)$** time.

Why this is called the discrete Fourier transform

- Real Fourier series
  - Given a real valued function f defined on $[0,2\pi]$ the Fourier series for f is given by $f(x)=a_0+a_1\cos(x)+a_2\cos(2x)+\ldots+a_m\cos(mx)+\ldots$ where

$$\boxed{a_m=\frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx}$$

  - is the component of f of frequency m
  - In signal processing and data compression one ignores all but the components with large a_m and there aren't many of these.

Why this is called the discrete Fourier transform

- Complex Fourier series
  - Given a function f defined on $[0,2\pi]$ the complex Fourier series for f is given by

    $f(z)=b_0+b_1\, e^{iz} + b_2\, e^{2iz} +\ldots+ b_m\, e^{miz} +\ldots$

    where

    $$\boxed{b_m= \frac{1}{2\pi}\int_0^{2\pi} f(z)\, e^{-miz}\, dz}$$

    is the component of f of frequency m
  - If we **discretize** this integral using values at n equally spaced points between 0 and 2π we get **$2\pi/n$ apart**

$$\boxed{\bar{b}_m = \frac{1}{n}\sum_{k=0}^{n-1} f_k\, e^{-2km i\pi/n} = \frac{1}{n}\sum_{k=0}^{n-1} f_k\ \omega^{-km} \text{ where } f_k=f(2k\pi/n)}$$

just like interpolation!

Beyond the Master Theorem

- Divide and conquer examples
  - Simple, randomized median algorithm
    - Expected **O(n)** time
  - Not so simple, deterministic median algorithm
    - Worst case **O(n)** time
  - Expected time analysis for Randomized QuickSort
    - Expected **O(n log n)** time

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Order problems: Find the kth largest

- Runtime models
  - Machine Instructions
  - Comparisons
- Maximum
  - **O(n)** time
  - **n-1** comparisons
- 2nd Largest
  - **O(n)** time
  - ? comparisons

Median Problem

- k^{th} largest for **k = n/2**
- Easily done in **O(n log n)** time with sorting
  - How can the problem be solved in **O(n)** time?

- Select(**k**, **n**) – find the **k**-th largest from a list of length **n**

Divide and Conquer

- $T(n) = n + T(\alpha n)$ for $\alpha < 1$
- Linear time solution

- Select algorithm – in linear time, reduce the problem from selecting the **k**-th largest of **n** to the **j**-th largest of α**n**, for $\alpha < 1$

Quick Select

```
QSelect($\mathbf{k}$, $\mathbf{S}$)
    Choose element $\mathbf{x}$ from $\mathbf{S}$
    $\mathbf{S_L} = \{\mathbf{y}$ in $\mathbf{S} \mid \mathbf{y} < \mathbf{x}\}$
    $\mathbf{S_E} = \{\mathbf{y}$ in $\mathbf{S} \mid \mathbf{y} = \mathbf{x}\}$
    $\mathbf{S_G} = \{\mathbf{y}$ in $\mathbf{S} \mid \mathbf{y} > \mathbf{x}\}$
    if $| \mathbf{S_L} | \geq \mathbf{k}$
        return QSelect($\mathbf{k}$, $\mathbf{S_L}$)
    else if $|\mathbf{S_L}| + |\mathbf{S_E}| \geq \mathbf{k}$
        return $\mathbf{y}$
    else
        return QSelect($\mathbf{k} - |\mathbf{S_L}| - |\mathbf{S_E}|$, $\mathbf{S_G}$)
```

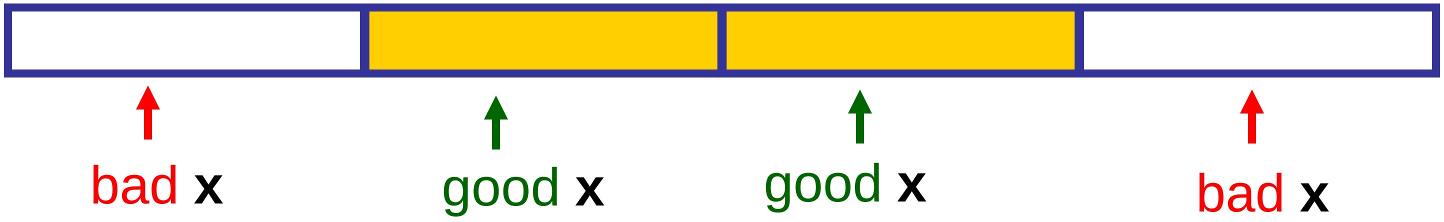
Implementing "Choose an element x"

- Ideally, we would choose an **x** in the middle, to reduce both sets in half and guarantee progress
- Method 1
  - Select an element at random
- Method 2
  - BFPRT Algorithm
  - Select an element by a complicated, but linear time method that guarantees a good split

Random Selection

Consider a call to QSelect(**k**, **S**), and let **S’** be the elements passed to the recursive call.

With probability at least $\frac{1}{2}$, $|S'| < \frac{3}{4}|S|$



elements of **S** listed in sorted order

$\Rightarrow$ On average only **2** recursive calls before the size of **S’** is at most **3n/4**

Expected runtime is O(n)

- Given **x**, one pass over **S** to determine S_L, S_E, and S_G and their sizes: **cn** time.
  - Expect **2cn** cost before size of **S'** drops to at most **3|S|/4**

- Let **T(n)** be the expected running time
- $T(n) \le T(3n/4) + 2cn$

$$\le 2cn + (3/4)\, 2cn + (3/4)^2\, 2cn + \ldots$$

$$\le 2cn\, (1 + (3/4) + (3/4)^2 + \ldots)$$

Making the algorithm deterministic

- In **O(n)** time, find an element that guarantees that the larger set in the split has size at most **$\frac{3}{4}$ n**

Blum-Floyd-Pratt-Rivest-Tarjan Algorithm

- Divide **S** into **n/5** sets of size **5**
- Sort each of these sets of size **5**
- Let **M** be the set of all medians of the sets of size **5**
- Let **x** be the median of **M**
- $S_L = \{y \text{ in } S \mid y < x\}$, $S_G = \{y \text{ in } S \mid y > x\}$
- Claim: $|S_L| < ¾ |S|$, $|S_G| < ¾ |S|$

BFPRT, Step 1: Construct sets of size 5, sort each set

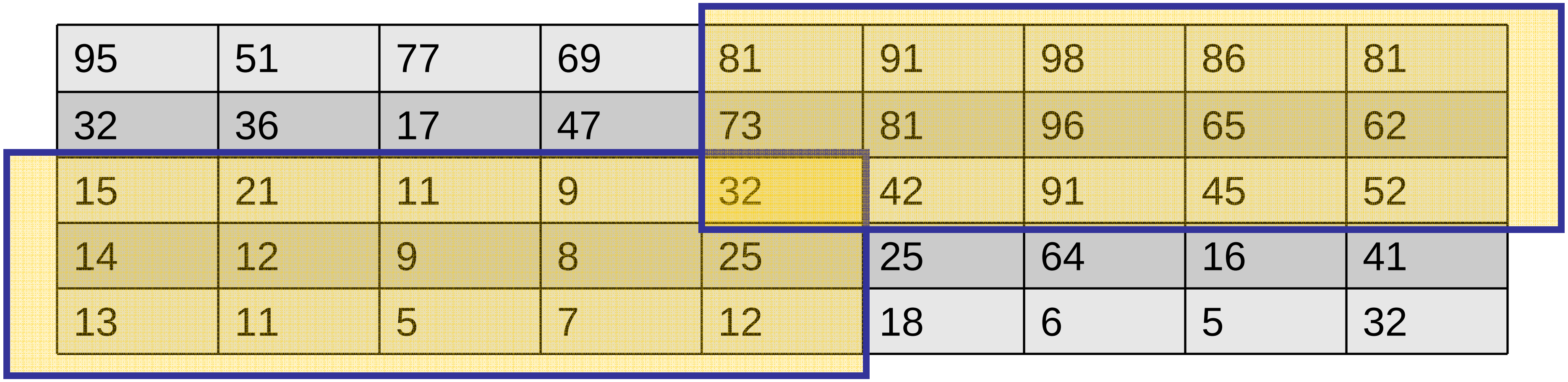
13, 15, 32, 14, 95, 5, 16, 45, 86, 65, 62, 41, 81, 52, 32, 32, 12, 73, 25, 81, 47, 8, 69, 9, 7, 81, 18, 25, 42, 91, 64, 98, 96, 91, 6, 51, 21, 12, 36, 11, 11, 9, 5, 17, 77

13	5	62	32	47	81	64	51	11
15	16	41	12	8	18	98	21	9
32	45	81	73	69	25	96	12	5
14	86	52	25	9	42	91	36	17
95	65	32	81	7	91	6	11	77

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

BFPRT, Step 2: Find median of column medians

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5



95	51	77	69	81	91	98	86	81
32	36	17	47	73	81	96	65	62
15	21	11	9	32	42	91	45	52
14	12	9	8	25	25	64	16	41
13	11	5	7	12	18	6	5	32

BFPRT Recurrence

- Sorting all **n/5** lists of size **5**
  - **c'n** time
- Finding median of set **M** of medians
  - Recursive computation: **T(n/5)**
- Computing sets S_L, S_E, S_G and **S'**
  - **c''n** time
- Solving selection problem on **S'**
  - Recursive computation: **T(3n/4)** since $|S'| \leq \frac{3}{4} n$

$T(n) \leq cn + T(n/5) + T(3n/4)$ is $O(n)$

- Key property
  - **$3/4 + 1/5 < 1$** (The sum is **19/20**)

- Sum of problem sizes decreases by **19/20** factor per level of recursion
- Overhead per level is linear in the sum of the problem sizes
  - Overhead decreases by **19/20** factor per level of recursion
  - Total overhead is linear (sum of geometric series with constant ratio and linear largest term)

Quick Sort

```
QuickSort(S)
    if S is empty, return
    Choose element x from S  “pivot”
    $S_L = \{y \text{ in } S \mid y < x\}$
    $S_E = \{y \text{ in } S \mid y = x\}$
    $S_G = \{y \text{ in } S \mid y > x\}$
    return [QuickSort($S_L$), $S_E$, QuickSort($S_G$)]
```

QuickSort

- Pivot Selection
  - Choose the median
    - $T(n) = T(n/2) + T(n/2) + cn$, $O(n \log n)$
  - Choose arbitrary element
    - Worst case – $O(n^2)$
    - Average case – $O(n \log n)$
  - Choose random pivot
    - Expected time – $O(n \log n)$

Expected run time for QuickSort: "Global analysis"

- Count comparisons
- a_i, a_j – elements in positions **i** and **j** in the **final** sorted list. p_{ij} the probability that a_i and a_j are compared
- Expected number of comparisons:

$$\sum_{i<j} p_{ij}$$

Lemma: $P_{ij} \leq 2/(j - i + 1)$

If **a_i** and **a_j** are compared then it must be during the call when they end up in different subproblems

- Before that, they aren't compared to each other
- After they aren't compared to each other

During this step they are only compared if one of them is the pivot

Since all elements between **a_i** and **a_j** are also in the subproblem this is **2** out of at least **$j\text{-}i\text{+}1$** choices

Average runtime is 2nln n

$$\sum_{i<j} p_{ij} \le \sum_{i<j} 2/(j-i+1)$$

write $j=k+i$

$$= 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} 1/(k+1)$$

$$\le 2\,(n-1)\,(H_n - 1)$$

where $H_n = 1+1/2+1/3+1/4+\ldots$
$= \ln n + O(1)$

$$\le 2n \ln n + O(n) \le 1.387 n \log_2 n$$

Divide and Conquer Summary

- Powerful technique, when applicable
- Divide large problem into a few smaller problems of the same type
- Smaller problems must be a constant factor smaller