CSE 521: Design & Analysis of Algorithms I

Dynamic Programming

Paul Beame

Dynamic Programming

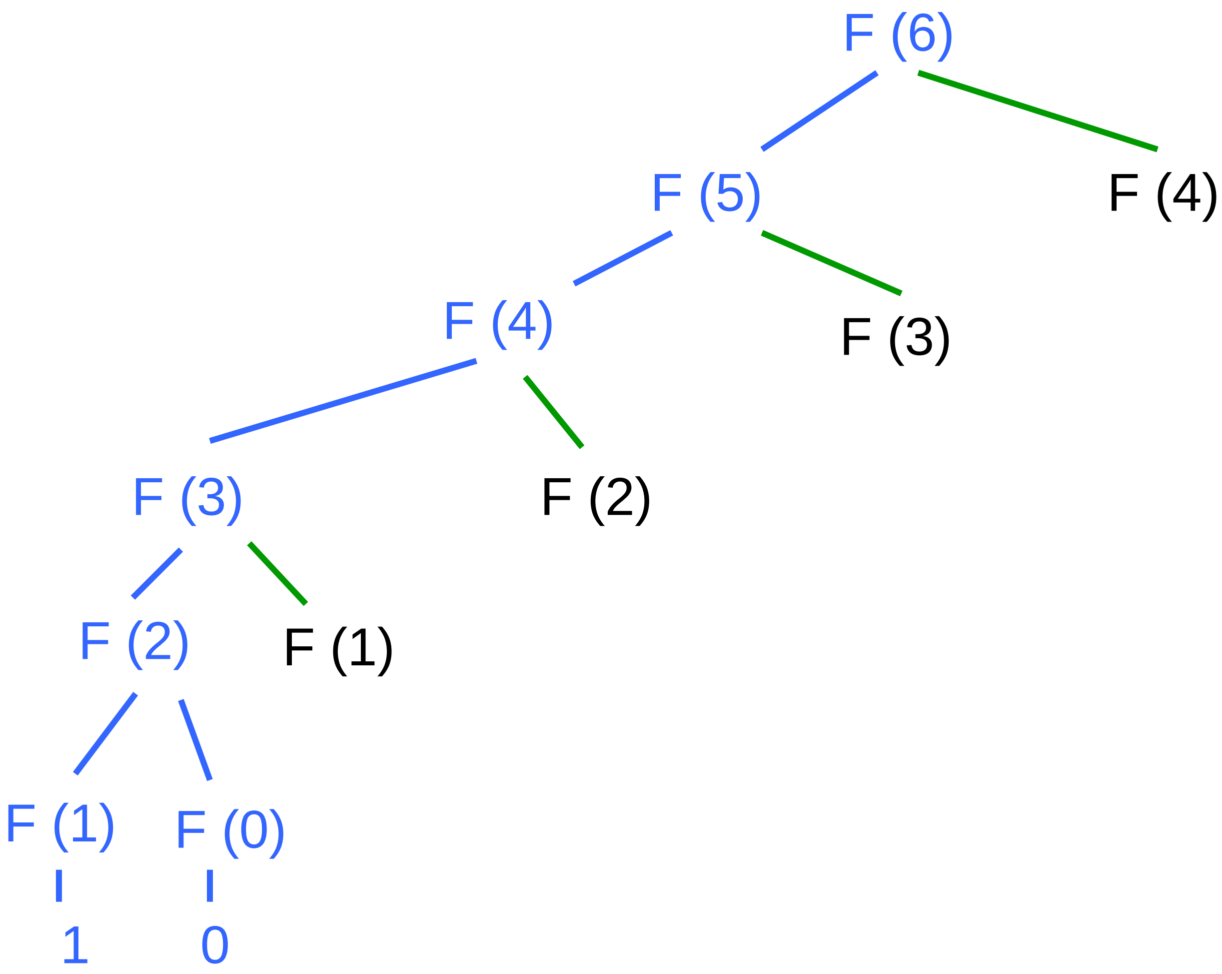
- Dynamic Programming
  - Give a solution of a problem using smaller sub-problems where the parameters of all the possible sub-problems are determined in advance
  - Useful when the same sub-problems show up again and again in the solution

A simple case:
Computing Fibonacci Numbers

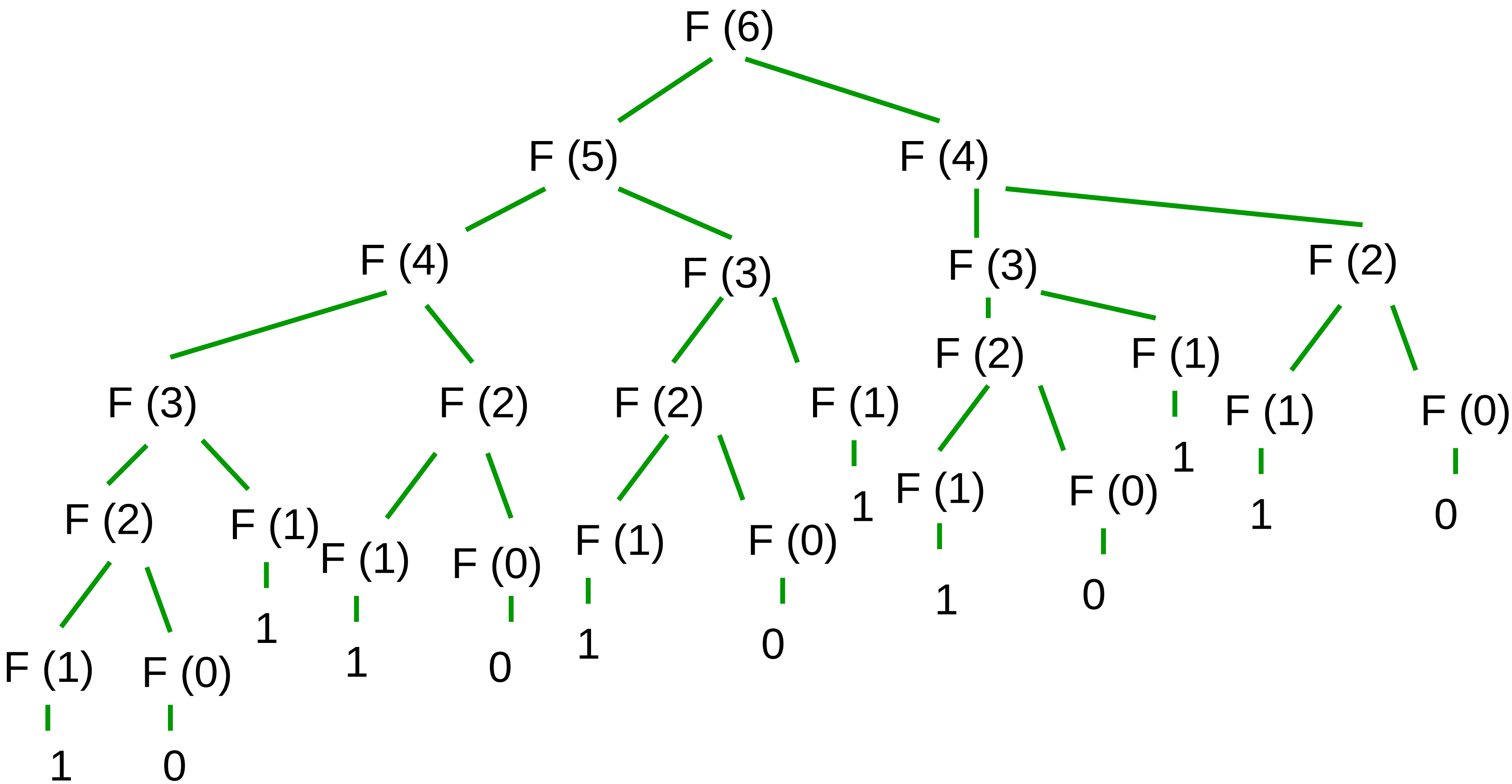
- Recall $F_n=F_{n-1}+F_{n-2}$ and $F_0=0$, $F_1=1$

- Recursive algorithm:
  - Fibo(**n**)
    if **n=0** then return(**0**)
    else if **n=1** then return(**1**)
    else return(Fibo(**n-1**)+Fibo(**n-2**))

Call tree - start



Full call tree



Memoization (Caching)

- Remember all values from previous recursive calls

- Before recursive call, test to see if value has already been computed

- Dynamic Programming
  - Convert memoized algorithm from a recursive one to an iterative one

Fibonacci
Dynamic Programming Version

- FiboDP(**n**):

  F[**0**]$\leftarrow$ **0**
  F[**1**] $\leftarrow$**1**
  for **i=2** to **n** do
  F[**i**]$\leftarrow$F[**i-1**]+F[**i-2**]
  endfor
  return(F[**n**])

Fibonacci: Space-Saving Dynamic Programming

- FiboDP(**n**):

    prev← **0**

    curr←**1**

    for **i=2** to **n** do

        temp←**curr**

        curr←**curr**+**prev**

        prev←**temp**

    endfor

    return(**curr**)

Dynamic Programming

- Useful when
  - same recursive sub-problems occur repeatedly
  - The solution to whole problem can be figured out with knowing the internal details of how the sub-problems are solved
    - principle of optimality

      "Optimal solutions to the sub-problems suffice for optimal solution to the whole problem"
  - Can anticipate the parameters of these recursive calls

Three Steps to Dynamic Programming

- Formulate the answer as a recurrence relation or recursive algorithm
- Show that the number of different values of parameters in the recursive calls is “small”
  - e.g., bounded by a low-degree polynomial
  - Can use memoization
- Specify an order of evaluation for the recurrence so that you already have the partial results ready when you need them.

Weighted Interval Scheduling

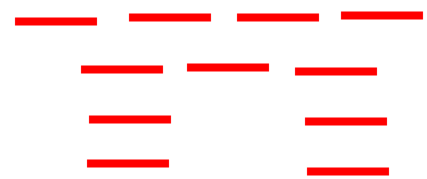
- Same problem as interval scheduling except that each request **i** also has an associated value or weight w_i
  - w_i might be
    - amount of money we get from renting out the resource for that time period
    - amount of time the resource is being used $w_i=f_i-s_i$
- Goal: Find compatible subset **S** of requests with maximum total weight

Greedy Algorithms for Weighted Interval Scheduling?

- No criterion seems to work
  - Earliest start time s_i
    - Doesn’t work

  - Shortest request time f_i-s_i
    - Doesn’t work

  - Fewest conflicts
    - Doesn’t work



  - Earliest finish fime f_i
    - Doesn’t work

  - Largest weight w_i
    - Doesn’t work

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Suppose that like ordinary interval scheduling we have first sorted the requests by finish time f_i so $f_1 \leq f_2 \leq \ldots \leq f_n$
- Say request **i** comes **before** request **j** if **i**< **j**
- For any request **j** let **p(j)** be
  - the largest-numbered request before **j** that is compatible with **j**
  - or **0** if no such request exists
- Therefore {**1**,…,**p(j)**} is precisely the set of requests before **j** that are compatible with **j**

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Two cases depending on whether an optimal solution **O** includes request **n**
  - If it **does** include request **n** then all other requests in **O** must be contained in {**1**,…,**p**(**n**)}
    - Not only that!
      - Any set of requests in {**1**,…,**p**(**n**)} will be compatible with request **n**
      - So in this case the optimal solution **O** must contain an optimal solution for {**1**,…,**p**(**n**)}
      - **"Principle of Optimality"**

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Two cases depending on whether an optimal solution **O** includes request **n**
  - If it **does not** include request **n** then all requests in **O** must be contained in {**1**,…, **n-1**}
    - Not only that!
      - The optimal solution **O** must contain an optimal solution for {**1**,…, **n-1**}
      - **"Principle of Optimality"**

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- All subproblems involve requests **{1,..,i}** for some **i**
- For **i=1,…,n** let **OPT(i)** be the weight of the optimal solution to the problem **{1,…,i}**
- The two cases give

  $\mathbf{OPT(n)=max[w_n+OPT(p(n)),OPT(n-1)]}$

- Also
  - $\mathbf{n \in O}$ iff $\mathbf{w_n+OPT(p(n))>OPT(n-1)}$

Towards Dynamic Programming: Step 1 – A Recursive Algorithm

- Sort requests and compute array **p[i]** for each **i=1,…,n**

```
ComputeOpt(n)
    if n=0 then return(0)
    else
        u←ComputeOpt(p[n])
        v←ComputeOpt(n-1)
        if w_n+u>v then return(w_n+u)
                    else return(v)
    endif
```

Towards Dynamic Programming: Step 2 – Small # of parameters

- ComputeOpt(**n**) can take exponential time in the worst case
  - **2^n** calls if **p(i)=i-1** for every **I**
- There are only **n** possible parameters to ComputeOpt
- Store these answers in an array **OPT[n]** and only recompute when necessary
  - Memoization
- Initialize **OPT[i]=0** for **i=1,…,n**

Dynamic Programming: Step 2 – Memoization

```
ComputeOpt(n)
    if n=0 then return(0)
    else
       u←MComputeOpt(p[n])
       v←MComputeOpt(n-1)
       if w_n+u>v then
           return(w_n+u)
       else return(v)
    endif
```

```
MComputeOpt(n)
       if OPT[n]=0 then
        v←ComputeOpt(n)
        OPT[n]←v
        return(v)
       else
        return(OPT[n])
       endif
```

Dynamic Programming Step 3: Iterative Solution

- The recursive calls for parameter **n** have parameter values **i** that are < **n**

```
IterativeComputeOpt(n)
   array OPT[0..n]
   OPT[0]←0
   for i=1 to n
        if w_i+OPT[p[i]] >OPT[i-1] then
                OPT[i] ←w_i+OPT[p[i]]
        else
                OPT[i] ←OPT[i-1]
        endif
   endfor
```

Producing the Solution

```
IterativeComputeOptSolution(n)
  array OPT[0..n], Used[1..n]
  OPT[0]←0
  for i=1 to n
    if wi+OPT[p[i]] >OPT[i-1] then
      OPT[i] ←wi+OPT[p[i]]
      Used[i]←1
    else
      OPT[i] ← OPT[i-1]
      Used[i] ←0
    endif
  endfor
```

```
FindSolution
i←n
S←∅
while i> 0 do
    if Used[i]=1 then
        S←S ∪ {i}
        i←p[i]
    else
        i←i-1
    endif
endwhile
```

Example

	1	2	3	4	5	6	7	8	9
s_i	4	2	6	8	11	15	11	12	18
f_i	7	9	10	13	14	17	18	19	20
w_i	3	7	4	5	3	2	7	7	2
p[i]									
OPT[i]									
Used[i]									

Example

	1	2	3	4	5	6	7	8	9
s_i	4	2	6	8	11	15	11	12	18
f_i	7	9	10	13	14	17	18	19	20
w_i	3	7	4	5	3	2	7	7	2
p[i]	0	0	0	1	3	5	3	3	7
OPT[i]									
Used[i]									

Example

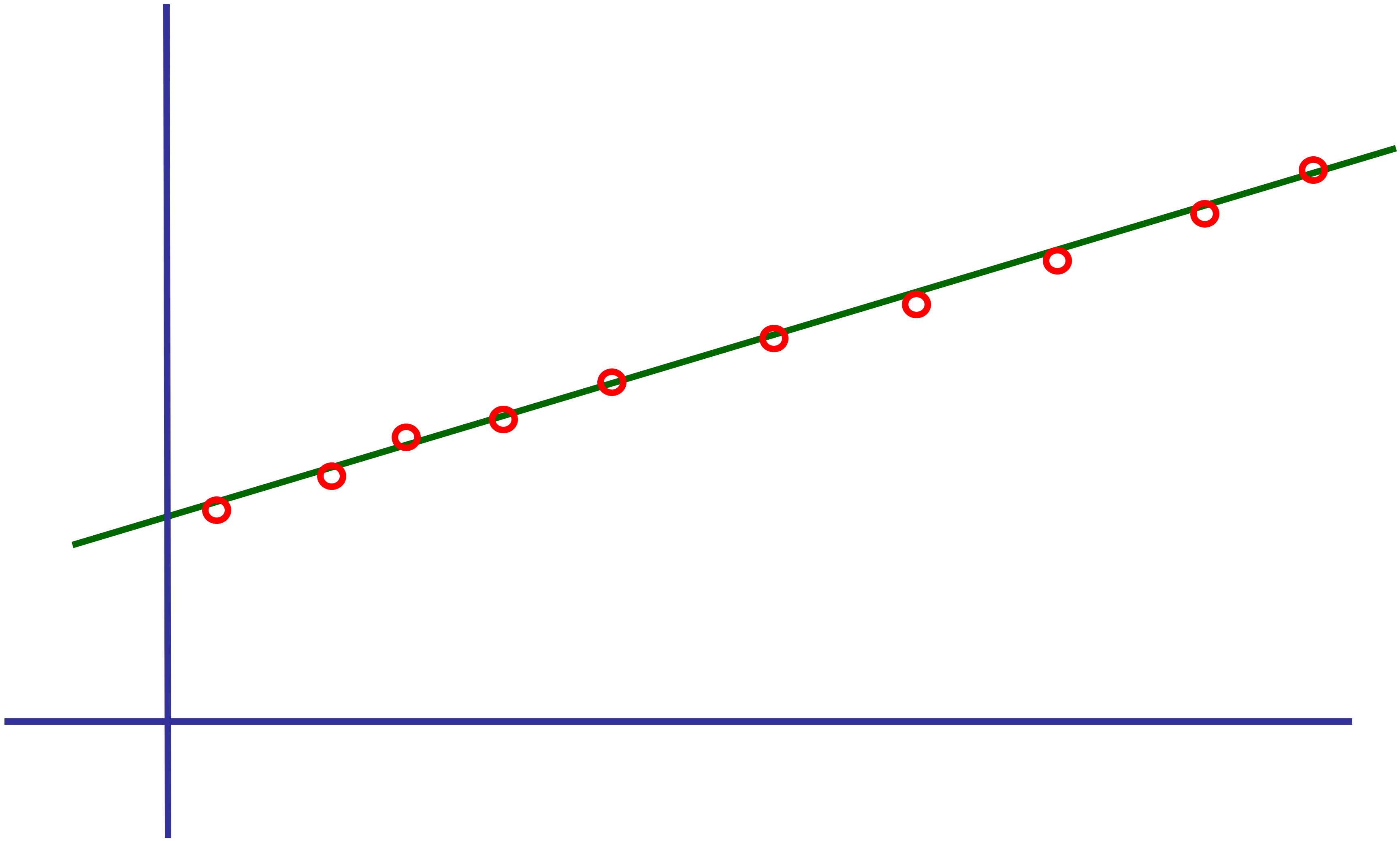
	1	2	3	4	5	6	7	8	9
s_i	4	2	6	8	11	15	11	12	18
f_i	7	9	10	13	14	17	18	19	20
w_i	3	7	4	5	3	2	7	7	2
p[i]	0	0	0	1	3	5	3	3	7
OPT[i]	3	7	7	8	10	12	14	14	16
Used[i]	1	1	0	1	1	1	1	0	1

S={9,7,2}

Segmented Least Squares

- Least Squares
  - Given a set **P** of **n** points in the plane $p_1=(x_1,y_1),\ldots,p_n=(x_n,y_n)$ with $x_1<\ldots<x_n$ determine a line **L** given by $y=ax+b$ that optimizes the totaled 'squared error'
    - $\text{Error}(L,P)=\sum_i(y_i-ax_i-b)^2$
  - A classic problem in statistics
  - Optimal solution is known (see text)
    - Call this line(**P**) and its error error(**P**)

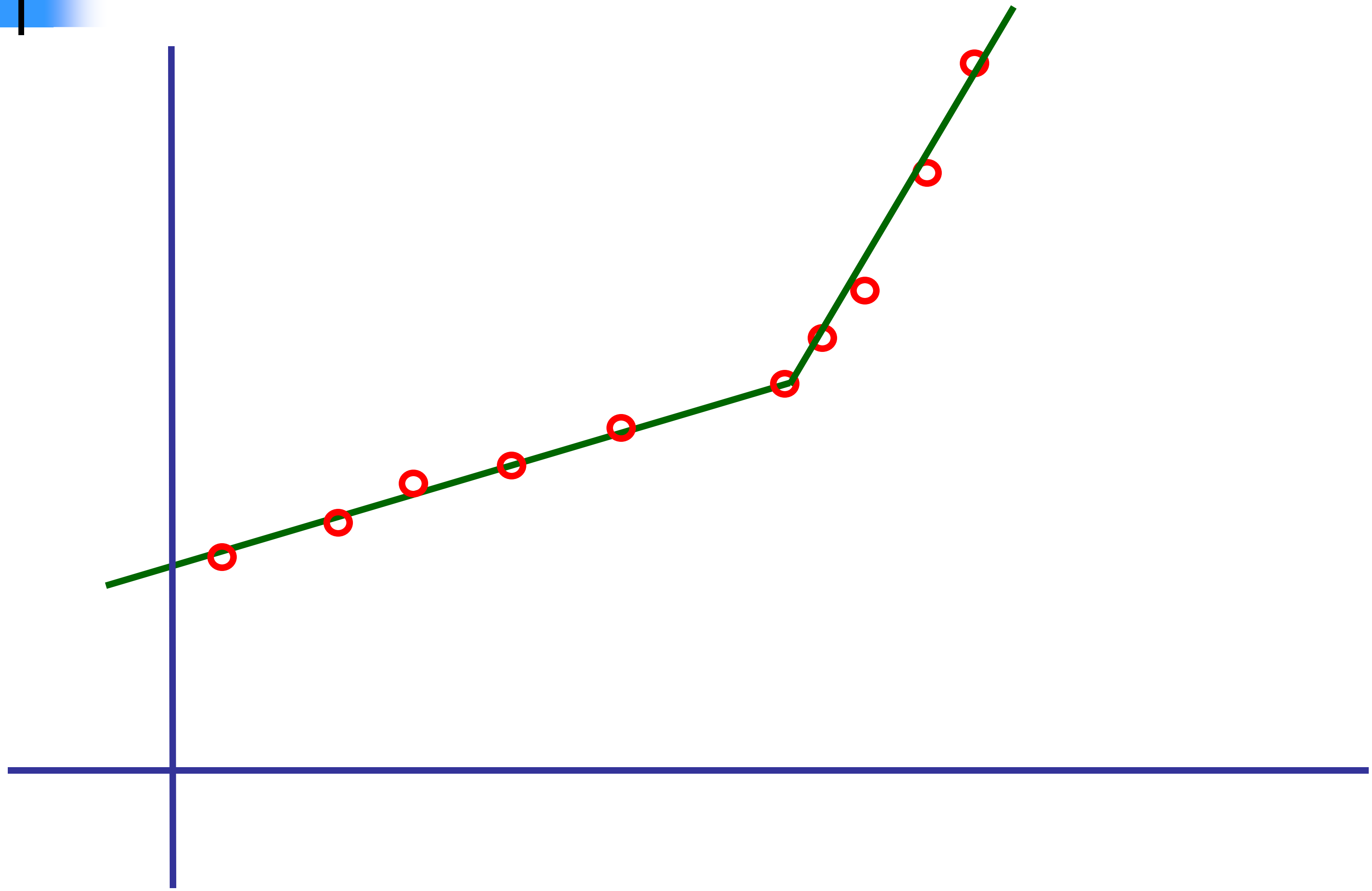
Least Squares



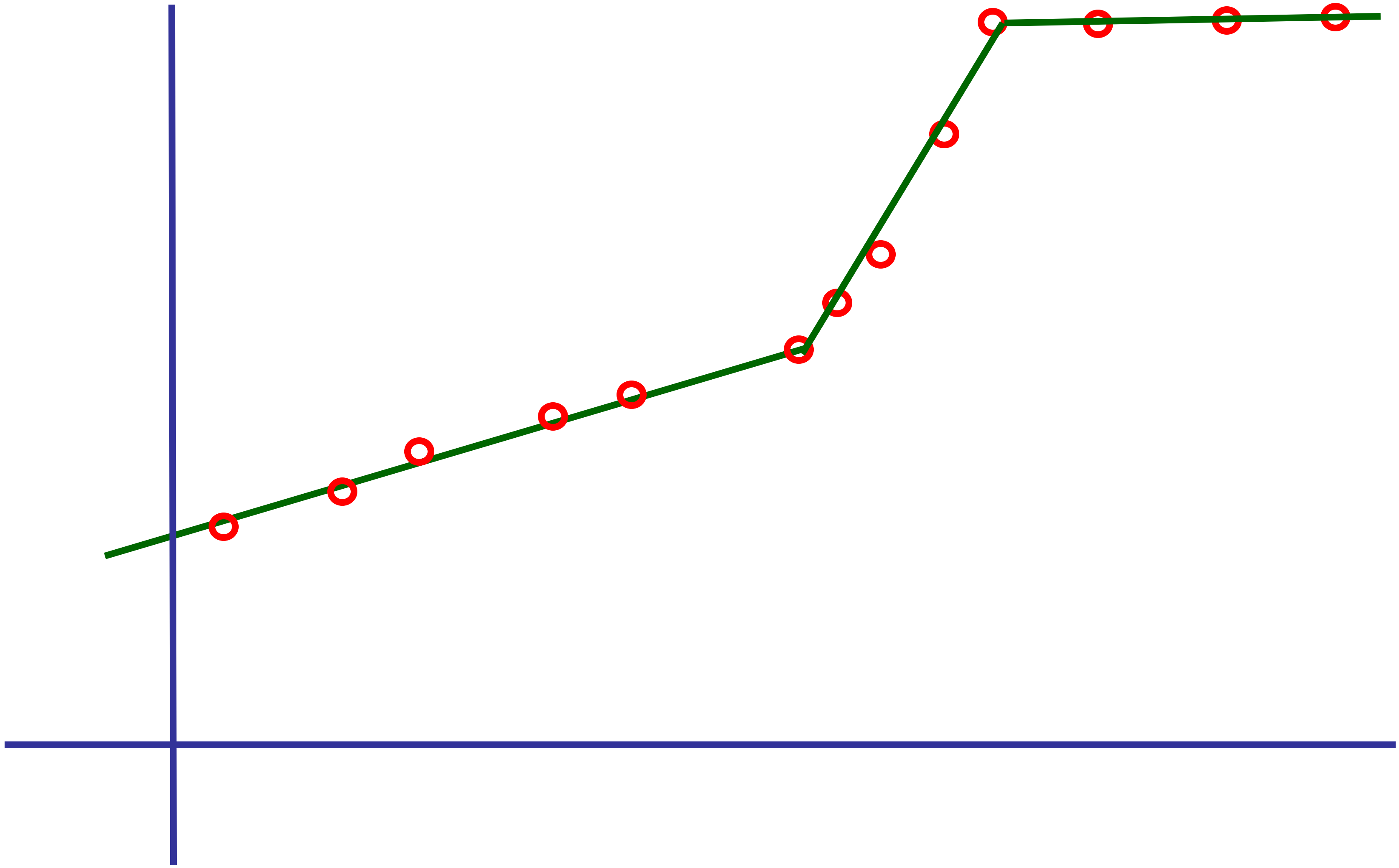
Segmented Least Squares

- What if data seems to follow a piece-wise linear model?

Segmented Least Squares



Segmented Least Squares



Segmented Least Squares

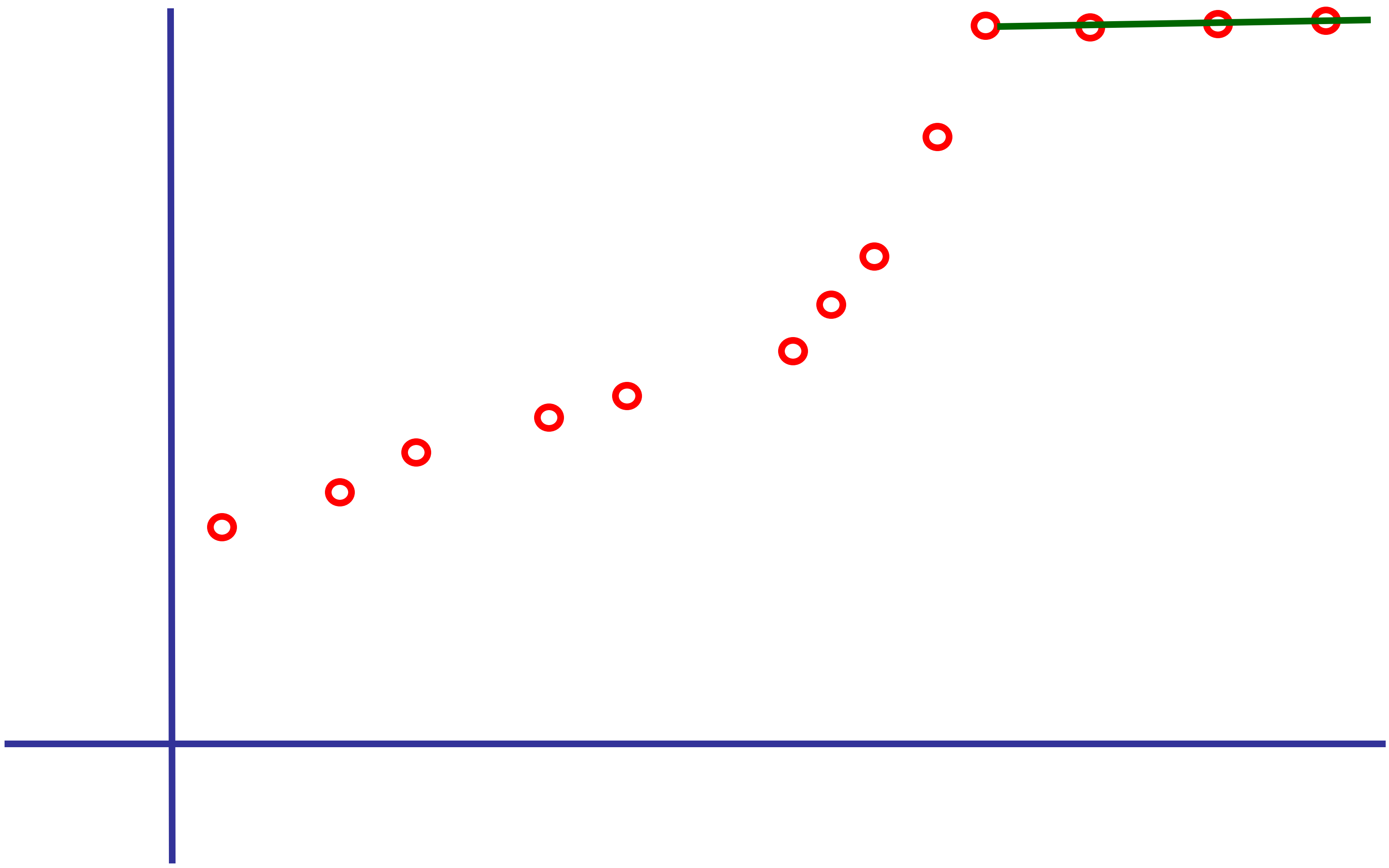
- What if data seems to follow a piece-wise linear model?
- Number of pieces to choose is not obvious
- If we chose **n-1** pieces we could fit with **0** error
  - Not fair
- Add a penalty of **C** times the number of pieces to the error to get a total penalty
- How do we compute a solution with the smallest possible total penalty?

Segmented Least Squares

- Recursive idea
  - If we knew the point $\mathbf{p_j}$ where the **last** line segment began then we could solve the problem optimally for points $\mathbf{p_1,\ldots,p_j}$ and combine that with the last segment to get a global optimal solution
    - Let OPT($\mathbf{i}$) be the optimal penalty for points $\{\mathbf{p_1},\ldots,\mathbf{p_i}\}$
    - Total penalty for this solution would be

$$\text{Error}(\{\mathbf{p_j},\ldots,\mathbf{p_n}\}) + \mathbf{C} + \text{OPT}(\mathbf{j\text{-}1})$$

Segmented Least Squares



Segmented Least Squares

- Recursive idea
  - We don't know which point is $\mathbf{p_j}$
    - But we do know that $\mathbf{1 \le j \le n}$
    - The optimal choice will simply be the best among these possibilities
  - Therefore

$$\mathrm{OPT}(\mathbf{n}) = \min_{\mathbf{1 \le j \le n}} \left\{ \mathrm{Error}(\{\mathbf{p_j}, \ldots, \mathbf{p_n}\}) + \mathbf{C} + \mathrm{OPT}(\mathbf{j-1}) \right\}$$

Dynamic Programming Solution

```
SegmentedLeastSquares(n)
  array OPT[0..n], Begin[1..n]
  OPT[0]←0
  for i=1 to n
     OPT[i]←Error{(p_1,…,p_i)}+C
     Begin[i]←1
     for j=2 to i-1
            e←Error{(p_j,…,p_i)}+C+OPT[j-1]
            if e <OPT[i] then
                 OPT[i] ←e
                 Begin[i]←j
            endif
     endfor
  endfor
  return(OPT[n])
```

```
FindSegments
i←n
S←∅
while i> 1 do
     compute Line({p_Begin[i],…,p_i})
     output (p_Begin[i],p_i), Line
     i←Begin[i]
endwhile
```

Knapsack (Subset-Sum) Problem

- Given:
  - integer **W** (knapsack size)
  - **n** object sizes $x_1, x_2, \ldots, x_n$
- Find:
  - Subset **S** of $\{1,\ldots,n\}$ such that $\sum_{i \in S} x_i \le W$ but $\sum_{i \in S} x_i$ is as large as possible

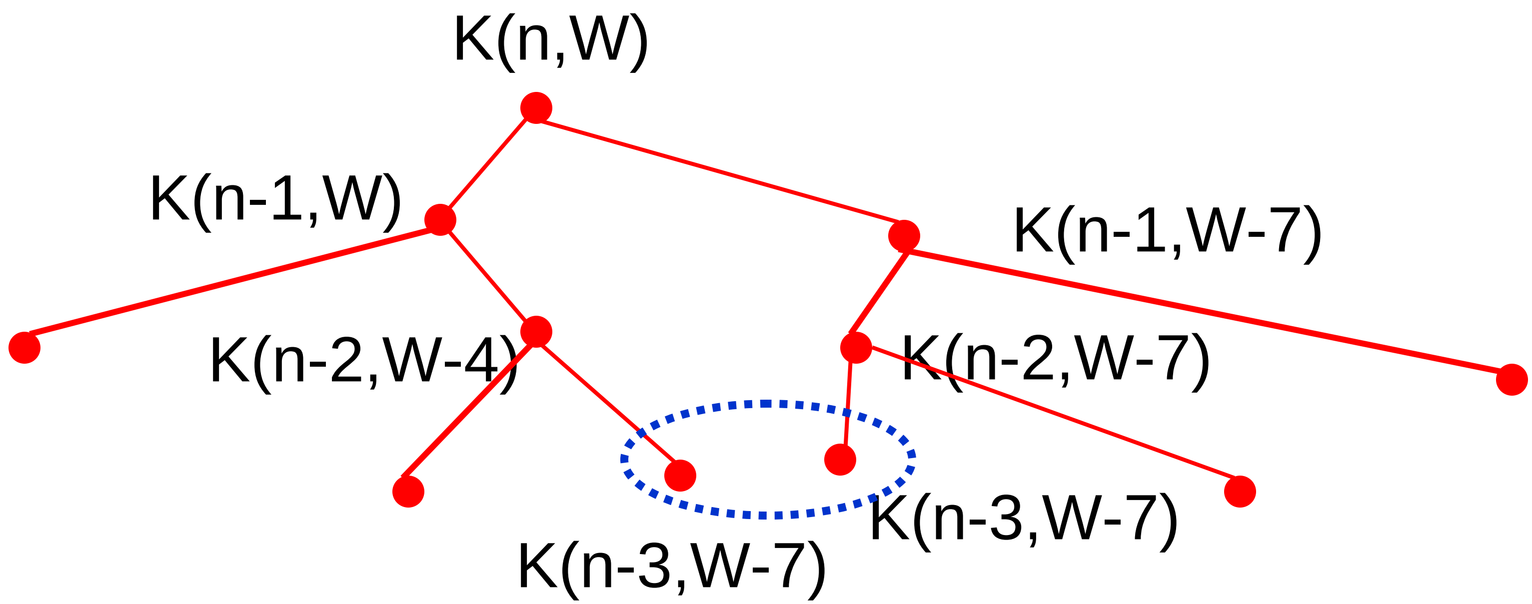
Recursive Algorithm

- Let **K(n,W)** denote the problem to solve for **W** and $x_1, x_2, \ldots, x_n$
- For **n>0**,
  - The optimal solution for **K(n,W)** is the better of the optimal solution for either

    K(n-1,W) or x_n+**K(n-1,W-**x_n**)**
  - For **n=0**
    - **K(0,W)** has a trivial solution of an empty set **S** with weight **0**

Recursive calls

- Recursive calls on list …,3, 4, 7



Common Sub-problems

- Only sub-problems are **K(i,w)** for
  - **i = 0,1,..., n**
  - **w = 0,1,..., W**
- Dynamic programming solution
  - Table entry for each **K(i,w)**
    - **OPT** - value of optimal soln for first **i** objects and weight **w**
    - **belong** flag - is x_i a part of this solution?
  - Initialize **OPT[0,w]** for **w=0,...,W**
  - Compute all **OPT[i,*]** from **OPT[i-1,*]** for **i>0**

Dynamic Knapsack Algorithm

for **w=0** to **W**; **OPT**[**0**,**w**] $\leftarrow$ **0**; end for
for **i**=**1** to **n** do
    for **w**=**0** to **W** do
        OPT[**i**,**w**]$\leftarrow$**OPT**[**i-1**,**w**]
        belong[**i**,**w**]$\leftarrow$**0**
        if **w** $\geq$ x_i then
            val $\leftarrow x_i$+**OPT**[**i**,**w**-x_i]
            if **val**>**OPT**[**i**,**w**] then
                OPT[**i**,**w**]$\leftarrow$**val**
                belong[**i**,**w**]$\leftarrow$**1**
    end for
end for
return(**OPT**[**n**,**W**])

Time O(nW)

Saving Space

- To compute the value OPT of the solution only need to keep the last two rows of OPT at each step

- What about determining the set S?
  - Follow the **belong** flags **O(n)** time
  - What about space?

Three Steps to Dynamic Programming

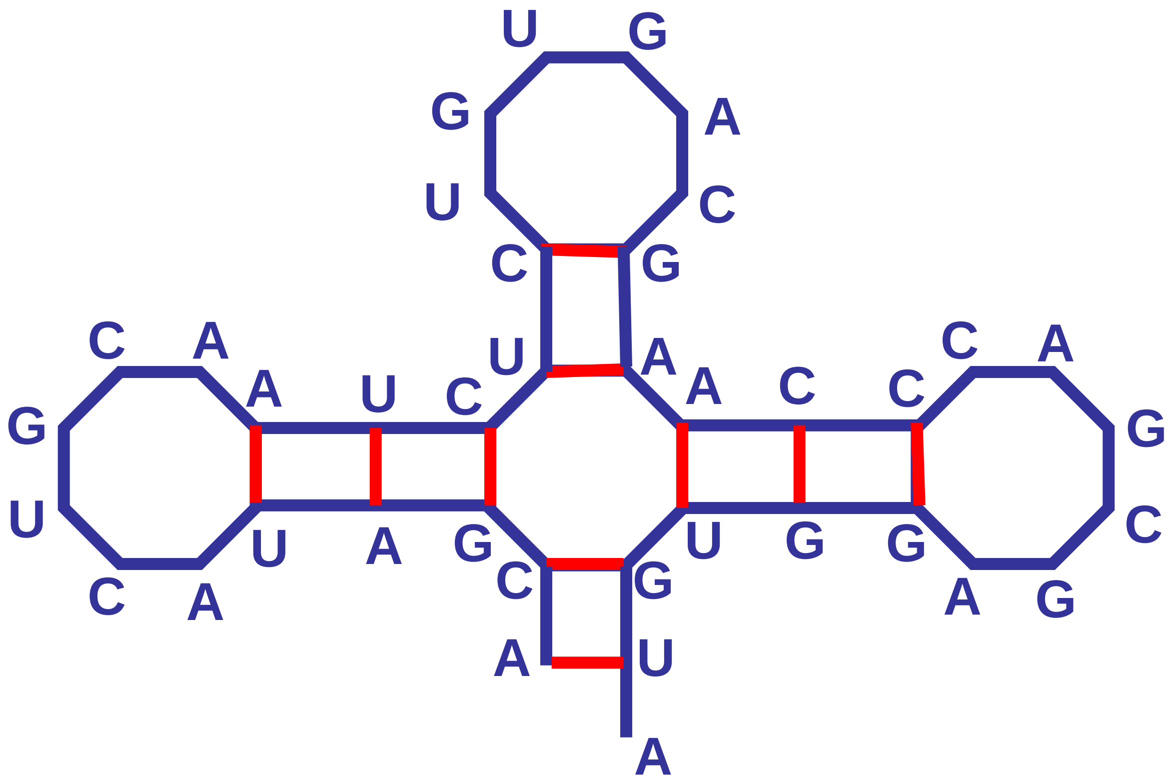
- Formulate the answer as a recurrence relation or recursive algorithm
- Show that the number of different values of parameters in the recursive algorithm is "small"
  - e.g., bounded by a low-degree polynomial
- Specify an order of evaluation for the recurrence so that you already have the partial results ready when you need them.

RNA Secondary Structure: Dynamic Programming on Intervals

- RNA: sequence of bases
  - String over alphabet {**A**, **C**, **G**, **U**}

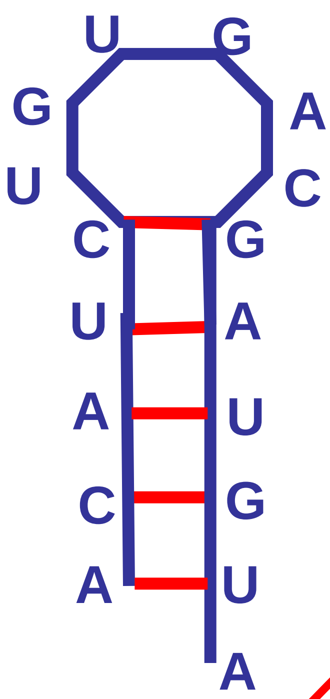
    U-G-U-A-C-C-G-G-U-A-G-U-A-C-A
- RNA folds and sticks to itself like a zipper
  - **A** bonds to **U**
  - **C** bonds to **G**
  - Bends can't be sharp
  - No twisting or criss-crossing
- How the bonds line up is called the RNA secondary structure

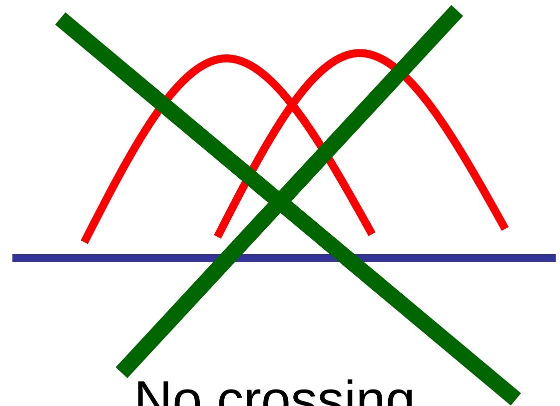
RNA Secondary Structure



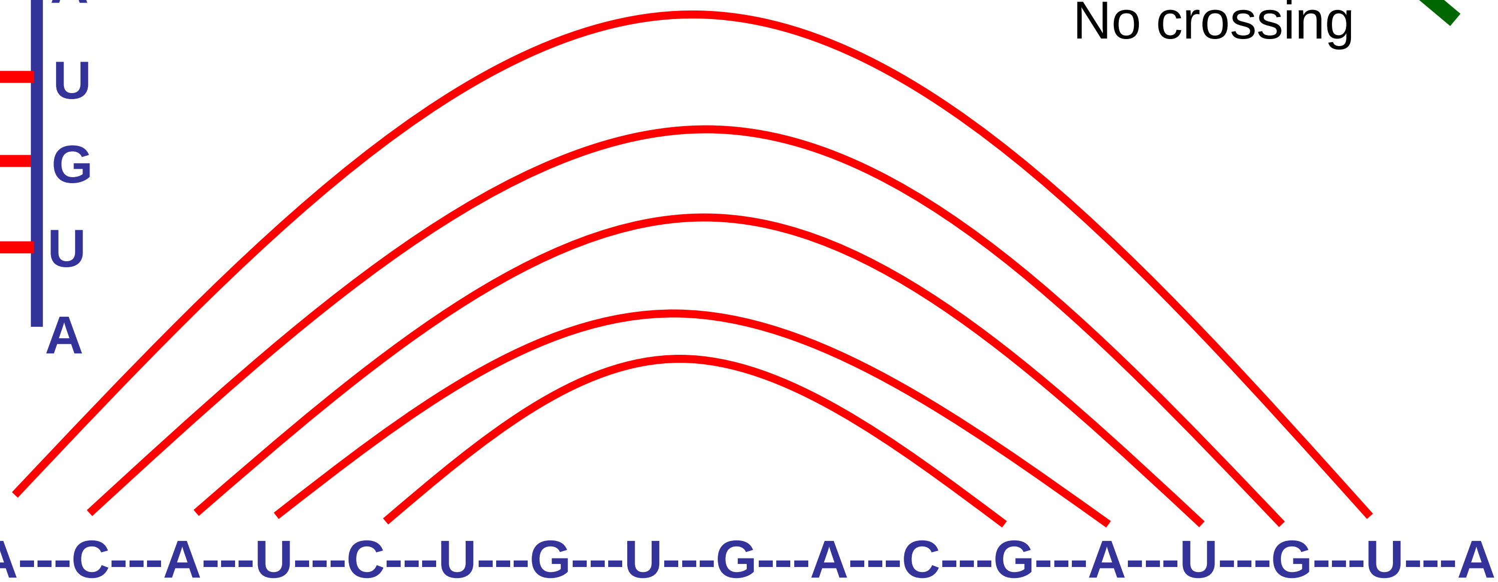
ACGAUACUGCAAUCUCUGUGACGAACCCAGCGAGGUGUA

Another view of RNA Secondary Structure





No crossing

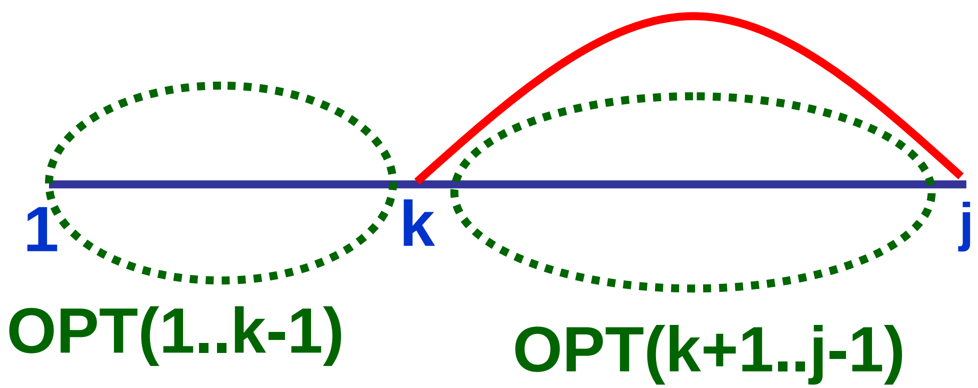


RNA Secondary Structure

- Input: String $x_1 \ldots x_n \in \{A,C,G,U\}^*$
- Output: Maximum size set **S** of pairs **(i,j)** such that
  - $\{x_i, x_j\} = \{A,U\}$ or $\{x_i, x_j\} = \{C,G\}$
  - The pairs in **S** form a matching
  - **i<j-4** (no sharp bends)
  - No crossing pairs
    - If **(i,j)** and **(k,l)** are in **S** then it is not the case that they cross as in **i<k<j<l**

Recursion Solution

- Try all possible matches for the last base



$$OPT(1..j)=MAX(OPT(1..j-1), 1+MAX_{k=1..j-5}\ (OPT(1..k-1)+OPT(k+1..j-1))$$

x_k matches x_j — Doesn't start at 1

General form:

$$OPT(i..j)=MAX(OPT(i..j-1),\ 1+MAX_{k=i..j-5}\ (OPT(i..k-1)+OPT(k+1..j-1)))$$

x_k matches x_j

RNA Secondary Structure

- 2D Array **OPT(i,j)** for **i≤j** represents optimal # of matches entirely for segment **i..j**
- For **j-i ≤4** set **OPT(i,j)=0** (no sharp bends)
- Then compute **OPT(i,j)** values when **j-i=5,6,...,n-1** in turn using recurrence.
- Return **OPT(1,n)**
- Total of **O(n^3)** time
- Can also record matches along the way to produce **S**
  - Algorithm is similar to the polynomial-time algorithm for Context-Free Languages based on Chomsky Normal Form
  - Both use dynamic programming over intervals

Sequence Alignment: Edit Distance

- Given:
  - Two strings of characters $A=a_1\ a_2\ \ldots\ a_n$ and $B=b_1\ b_2\ \ldots\ b_m$
- Find:
  - The minimum number of edit steps needed to transform **A** into **B** where an edit can be:
  - insert a single character
  - delete a single character
  - substitute one character by another

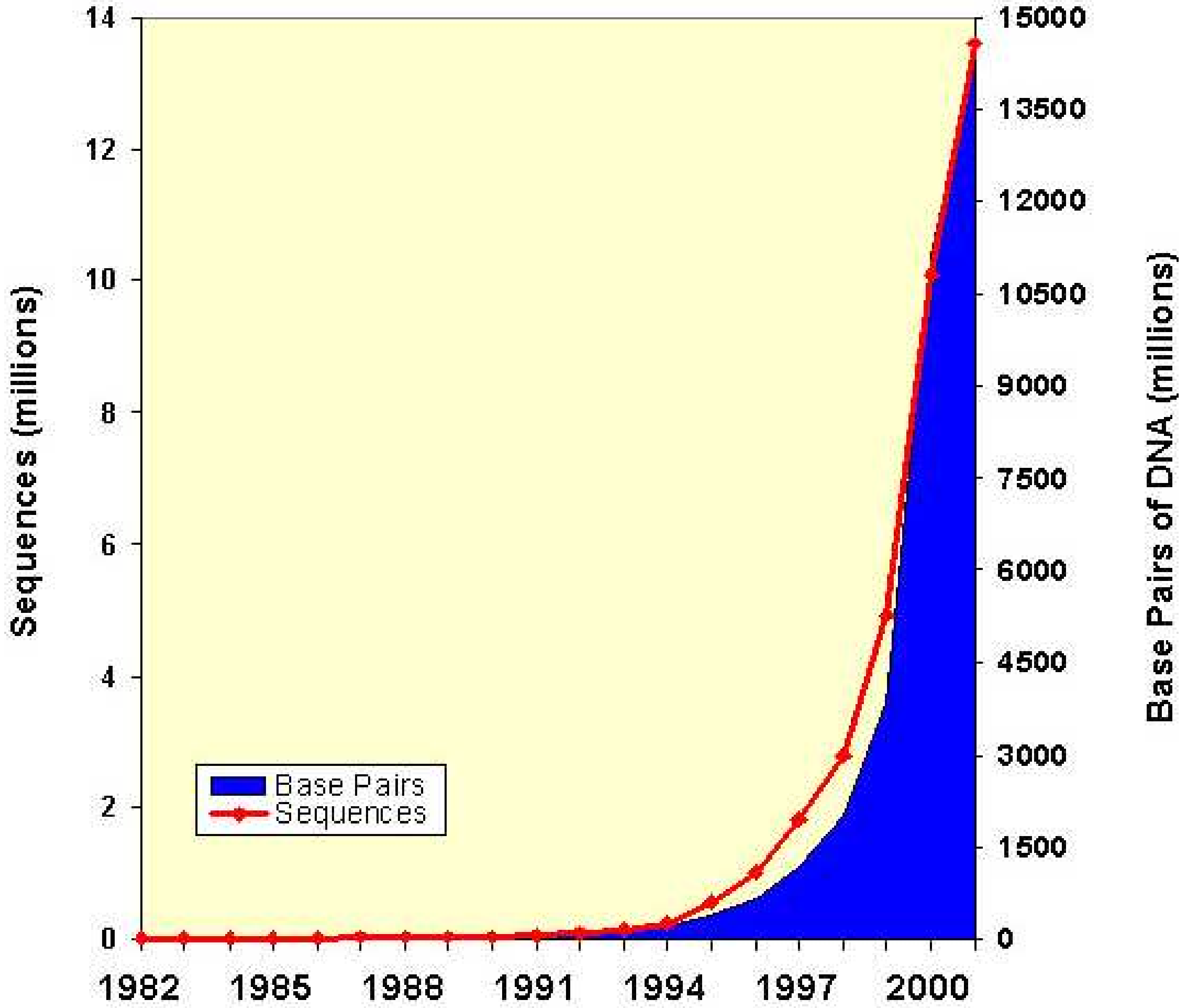
Sequence Alignment vs Edit Distance

- Sequence Alignment
  - Insert corresponds to aligning with a "**–**" in the first string
    - Cost δ (in our case **1**)
  - Delete corresponds to aligning with a "**–**" in the second string
    - Cost δ (in our case **1**)
  - Replacement of an **a** by a **b** corresponds to a mismatch
    - Cost α_{ab} (in our case **1** if $a \neq b$ and **0** if $a = b$)
- In Computational Biology this alignment algorithm is attributed to Smith & Waterman

Applications

- "diff" utility – where do two files differ
- Version control & patch distribution – save/send only changes
- Molecular biology
  - Similar sequences often have similar origin and function
  - Similarity often recognizable despite millions or billions of years of evolutionary divergence

Growth of GenBank



Recursive Solution

- Sub-problems: Edit distance problems for **all prefixes** of **A** and **B** that don't include all of both **A** and **B**

- Let **D(i,j)** be the number of edits required to transform $a_1\ a_2\ \ldots\ a_i$ into $b_1\ b_2\ \ldots\ b_j$

- Clearly **D(0,0)=0**

Computing D(n,m)

- Imagine how best sequence handles the last characters a_n and b_m
- If best sequence of operations
  - deletes a_n then $D(n,m)=D(n-1,m)+1$
  - inserts b_m then $D(n,m)=D(n,m-1)+1$
  - replaces a_n by b_m then $D(n,m)=D(n-1,m-1)+1$
  - matches a_n and b_m then $D(n,m)=D(n-1,m-1)$

Recursive algorithm D(n,m)

```
if  n=0  then
    return (m)
elseif  m=0  then
    return(n)
else
    if  a_n=b_m  then
        replace-cost ← 0
    else                        } cost of substitution of a_n by b_m (if used)
        replace-cost ← 1
    endif
    return(min{ D(n-1, m) + 1,
                D(n, m-1) +1,
                D(n-1, m-1) + replace-cost})
```

Dynamic Programming

for **j** = **0** to **m**; **D**(**0**,**j**) ← **j**; endfor

for **i** = **1** to **n**; **D**(**i**,**0**) ← **i**; endfor

for **i** = **1** to **n**

    for **j** = **1** to **m**

        if $a_i = b_j$ then

            replace-cost ← **0**

        else

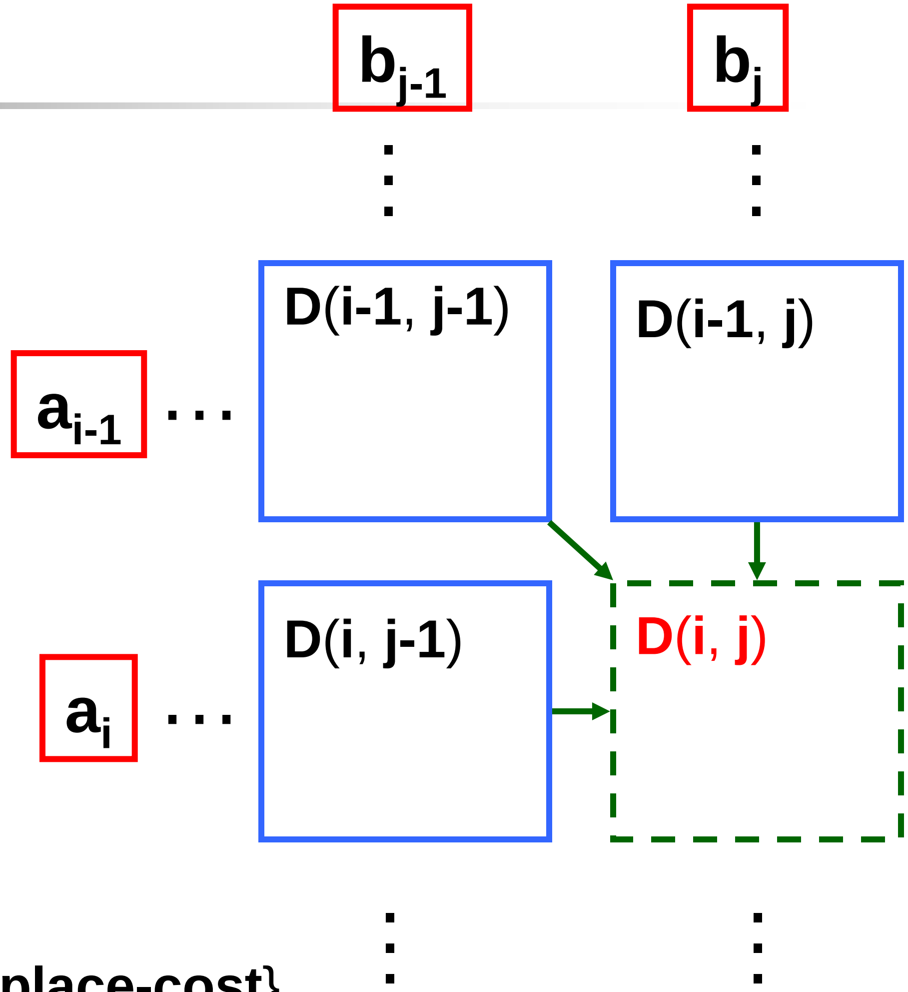
            replace-cost ← **1**

        endif

        D(**i**,**j**) ← **min** { **D**(**i-1**, **j**) + **1**,
                                D(**i**, **j-1**) + **1**,
                                D(**i-1**, **j-1**) + **replace-cost**}

    endfor

endfor



Example run with AGACATTG and GAGTTA

			A	G	A	C	A	T	T	G
		0	1	2	3	4	5	6	7	8
	0									
G	1									
A	2									
G	3									
T	4									
T	5									
A	6									

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	1	2	3	4	5	6	7	8
G	1	1	1	2	3	4	5	6	7
A	2	1	2	1	2	3	4	5	6
G	3	2	1	2	2	3	4	5	5
T	4	3	2	2	3	3	3	4	5
T	5	4	3	3	3	4	3	3	4
A	6	5	4	3	4	3	4	4	4

Example run with AGACATTG and GAGTTA

		A	G	A	C	A	T	T	G
	0	**1**	2	3	4	5	6	7	8
G	1	1	**1**	2	3	4	5	6	7
A	2	1	2	**1**	2	3	4	5	6
G	3	2	1	2	**2**	**3**	4	5	5
T	4	3	2	2	3	3	**3**	4	5
T	5	4	3	3	3	4	3	**3**	4
A	6	5	4	3	4	3	4	4	**4**

Reading off the operations

- Follow the sequence and use each color of arrow to tell you what operation was performed.
- From the operations can derive an optimal alignment

A G A C A T T G
_ G A G _ T T A

Saving Space

- To compute the distance values we only need the last two rows (or columns)
  - **O**(min(**m**,**n**)) space
- To compute the alignment/sequence of operations
  - seem to need to store all **O**(**mn**) pointers/arrow colors
  - Nifty divide and conquer variant that allows one to do this in **O**(min(**m**,**n**)) space and obtain **O**(**m+n**) time
  - In practice the algorithm is usually run on smaller chunks of a large string, e.g. **m** and **n** are lengths of genes so a few thousand characters
  - Full alignments only required for sequences with good scores
    - Researchers want all alignments that are close to optimal
    - Basic algorithm is run since the whole table of pointers (**2** bits each) will fit in RAM
  - Ideas are neat, though

Saving space

- Alignment corresponds to a path through the table from lower right to upper left
  - Must pass through the middle column
- Recursively compute the entries for the middle column from the left
  - If we knew the cost of completing each then we could figure out where the path crossed
  - Problem
    - There are **n** possible strings to start from.
  - Solution
    - **Recursively calculate the right half costs for each entry in this column using alignments starting at the other ends of the two input strings!**
  - Can reuse the storage on the left when solving the right hand problem

Shortest paths with negative cost edges (Bellman-Ford)

- Dijsktra's algorithm failed with negative-cost edges
  - What can we do in this case?
  - Negative-cost cycles could result in shortest paths with length **-∞**
- Suppose no negative-cost cycles in G
  - Shortest path from **s** to **t** has at most **n-1** edges
    - If not, there would be a repeated vertex which would create a cycle that could be removed since cycle can't have –ve cost

Shortest paths with negative cost edges (Bellman-Ford)

- We want to grow paths from **s** to **t** based on the # of edges in the path
- Let Cost(**s**,**t**,**i**)=cost of minimum-length path from **s** to **t** using up to **i** hops.
  - $\text{Cost}(\mathbf{v},\mathbf{t},\mathbf{0}) = \begin{cases} \mathbf{0} \text{ if } \mathbf{v}=\mathbf{t} \\ \infty \text{ otherwise} \end{cases}$
  - $\text{Cost}(\mathbf{v},\mathbf{t},\mathbf{i}) = \min\{\text{Cost}(\mathbf{v},\mathbf{t},\mathbf{i-1}), \min_{(\mathbf{v},\mathbf{w}) \in \mathbf{E}}(\mathbf{c}_{\mathbf{vw}} + \text{Cost}(\mathbf{w},\mathbf{t},\mathbf{i-1}))\}$

Bellman-Ford

- Observe that the recursion for Cost(**s**,**t**,**i**) doesn't change **t**
  - Only store an entry for each **v** and **i**
    - Termed OPT(**v**,**i**) in the text
- Also observe that to compute OPT(*,i) we only need OPT(*,i-1)
  - Can store a current and previous copy in O(**n**) space.

Bellman-Ford

ShortestPath(G,s,t)

for all **v**∈**V**

OPT[**v**]←∞

OPT[**t**]←**0**

for **i**=**1** to **n-1** do

for all **v**∈**V** do

OPT'[**v**]←$\min_{(v,w)\in E}$ (c_{vw}+**OPT**[**w**])

for all **v**∈**V** do

OPT[**v**]←min(**OPT'**[**v**],**OPT**[**v**])

return **OPT**[**s**]

O(mn) time

Negative cycles

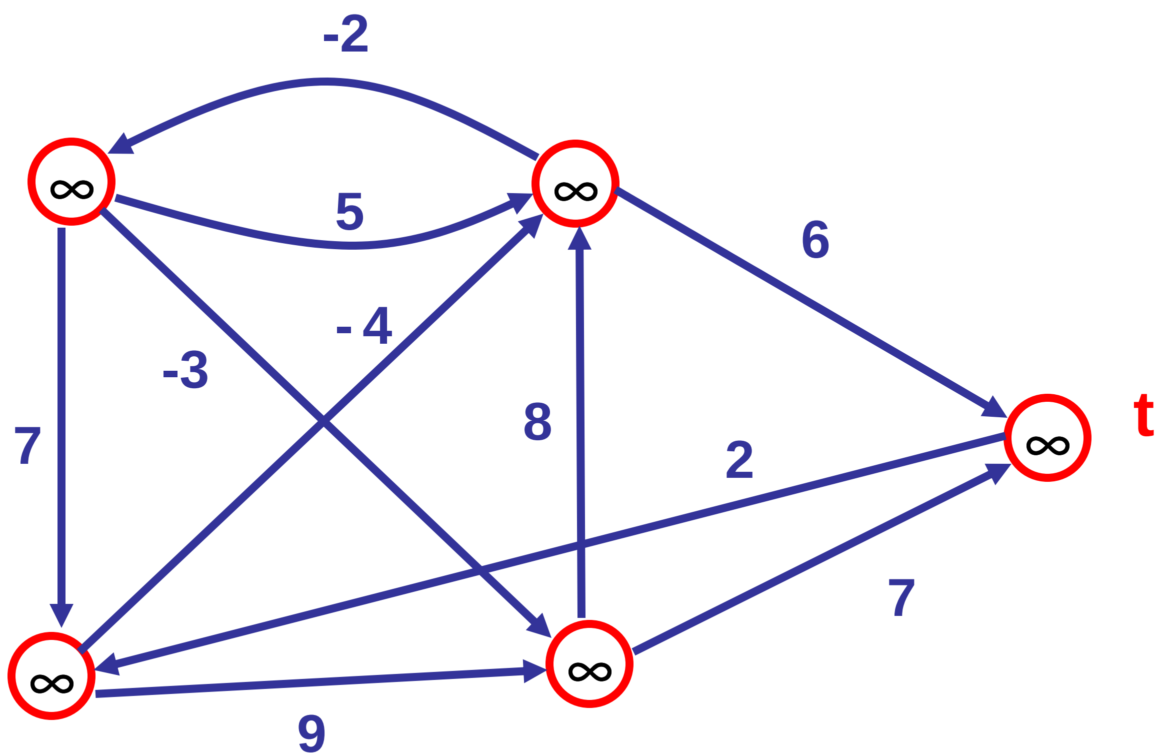
- Claim: There is a negative-cost cycle that can reach t iff for some vertex $\mathbf{v} \in \mathbf{V}$, Cost(**v**,**t**,**n**)<Cost(**v**,**t**,**n-1**)
- Proof:
  - We already know that if there aren't any then we only need paths of length up to **n-1**
  - For the other direction
    - The recurrence computes **Cost**(**v**,**t**,**i**) correctly for **any** number of hops **i**
    - The recurrence reaches a fixed point if for every $\mathbf{v} \in \mathbf{V}$, **Cost(v,t,i)=Cost(v,t,i-1)**
    - A negative-cost cycle means that eventually some **Cost(v,t,i)** gets smaller than any given bound
      - Can't have a –ve cost cycle if for every $\mathbf{v} \in \mathbf{V}$, **Cost(v,t,n)=Cost(v,t,n-1)**

Last details

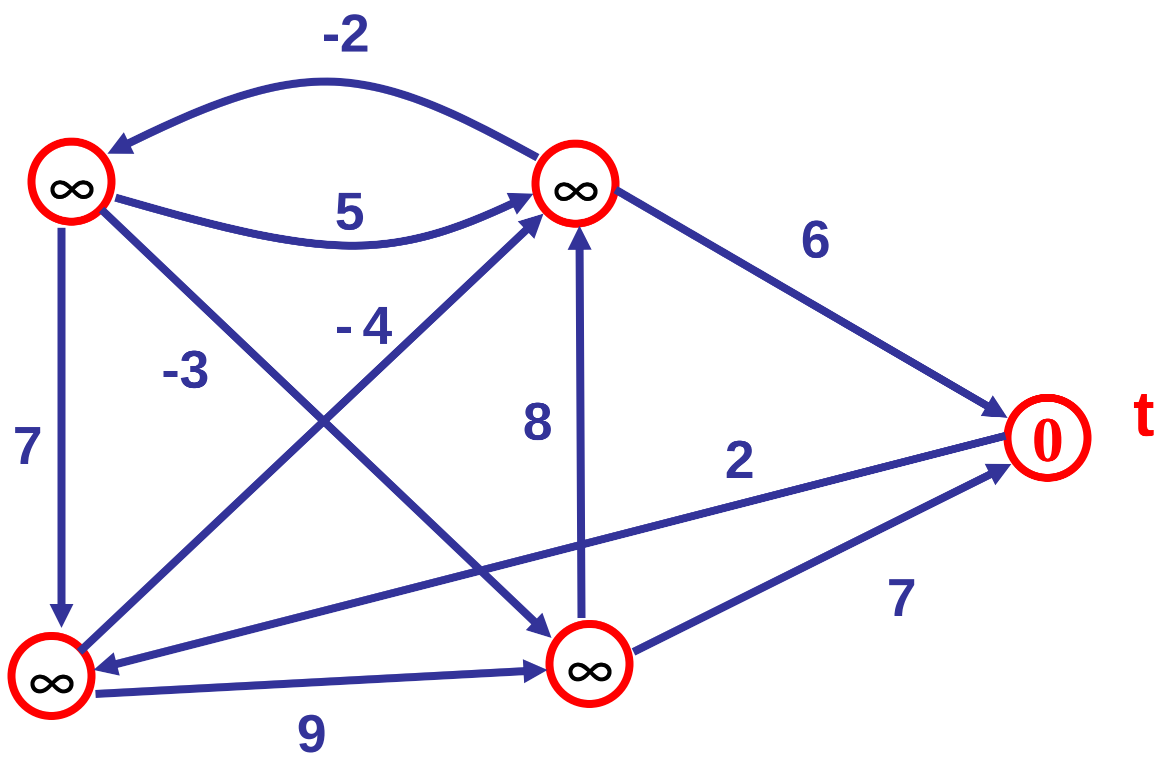
- Can run algorithm and stop early if the OPT and OPT' arrays are ever equal
  - Even better, one can update only neighbors **v** of vertices **w** with OPT'[**w**]≠OPT[**w**]
- Can store a successor pointer when we compute OPT
  - Homework assignment

- By running for step **n** we can find some vertex **v** on a negative cycle and use the successor pointers to find the cycle

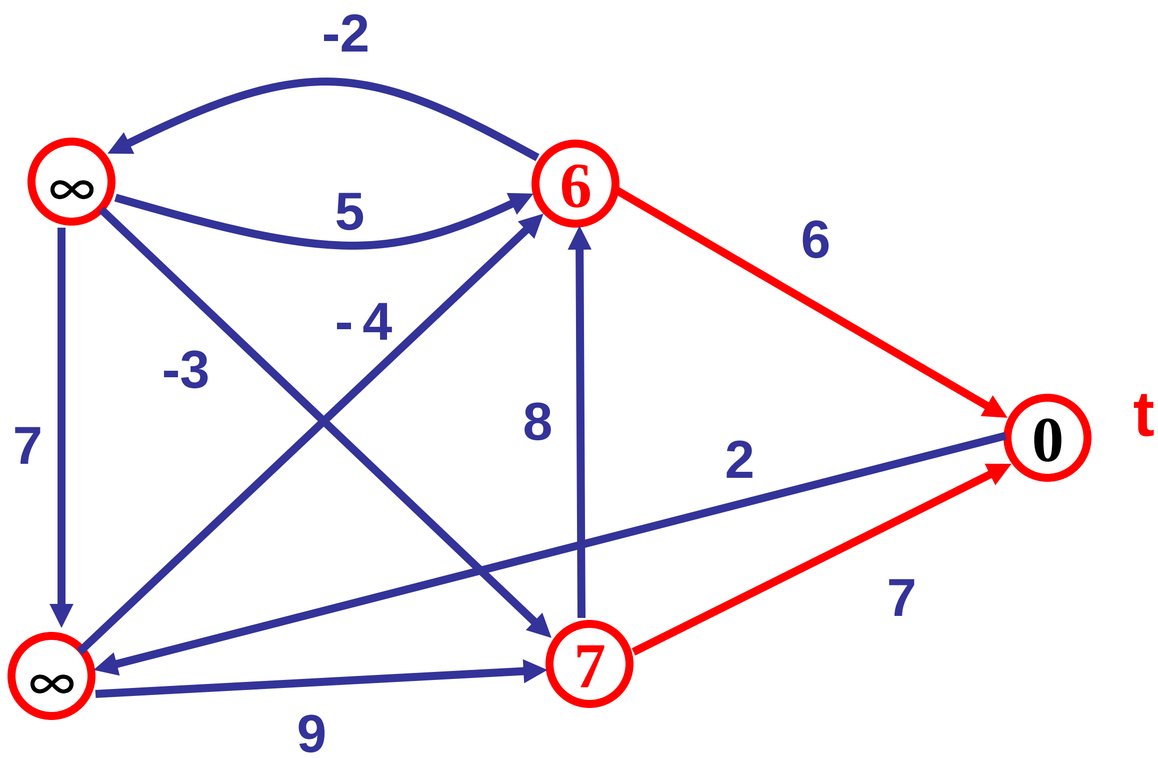
Bellman-Ford



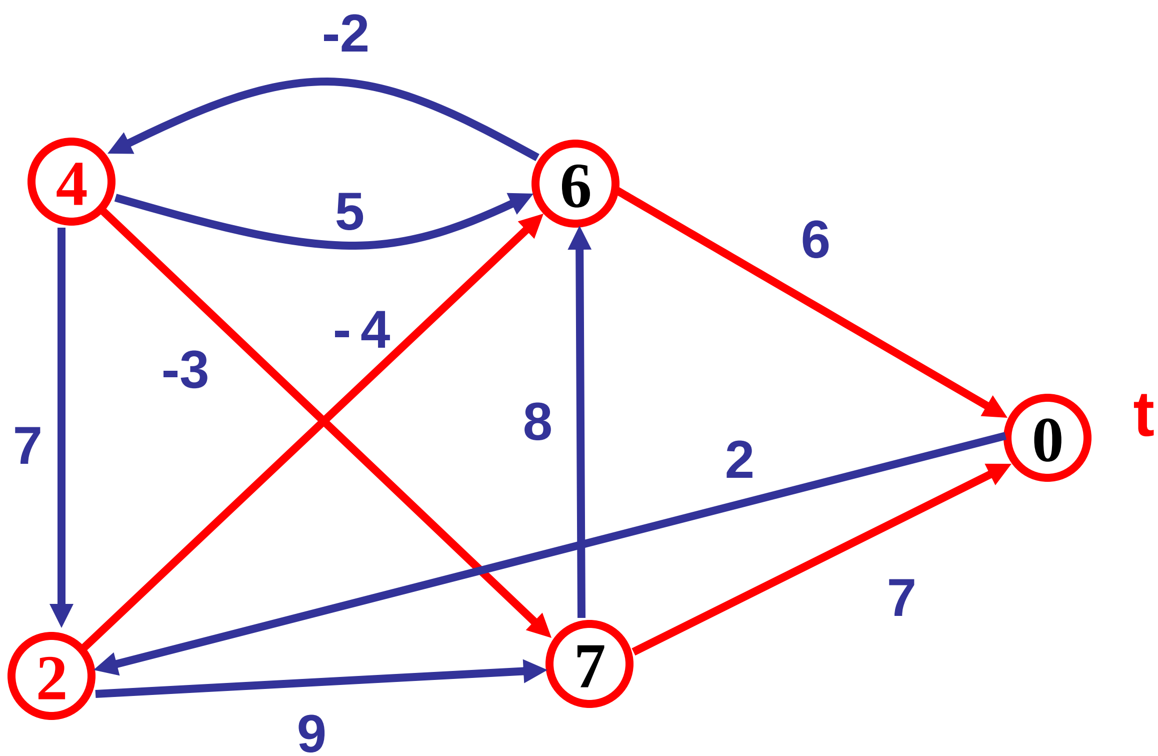
Bellman-Ford



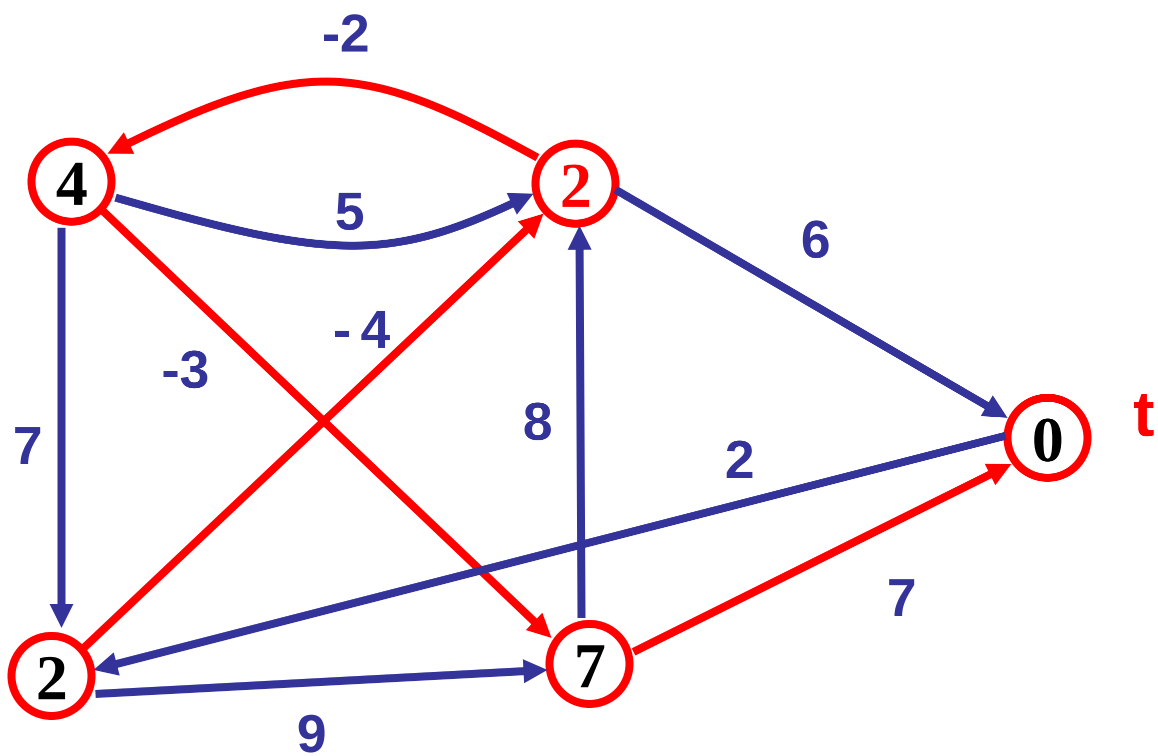
Bellman-Ford



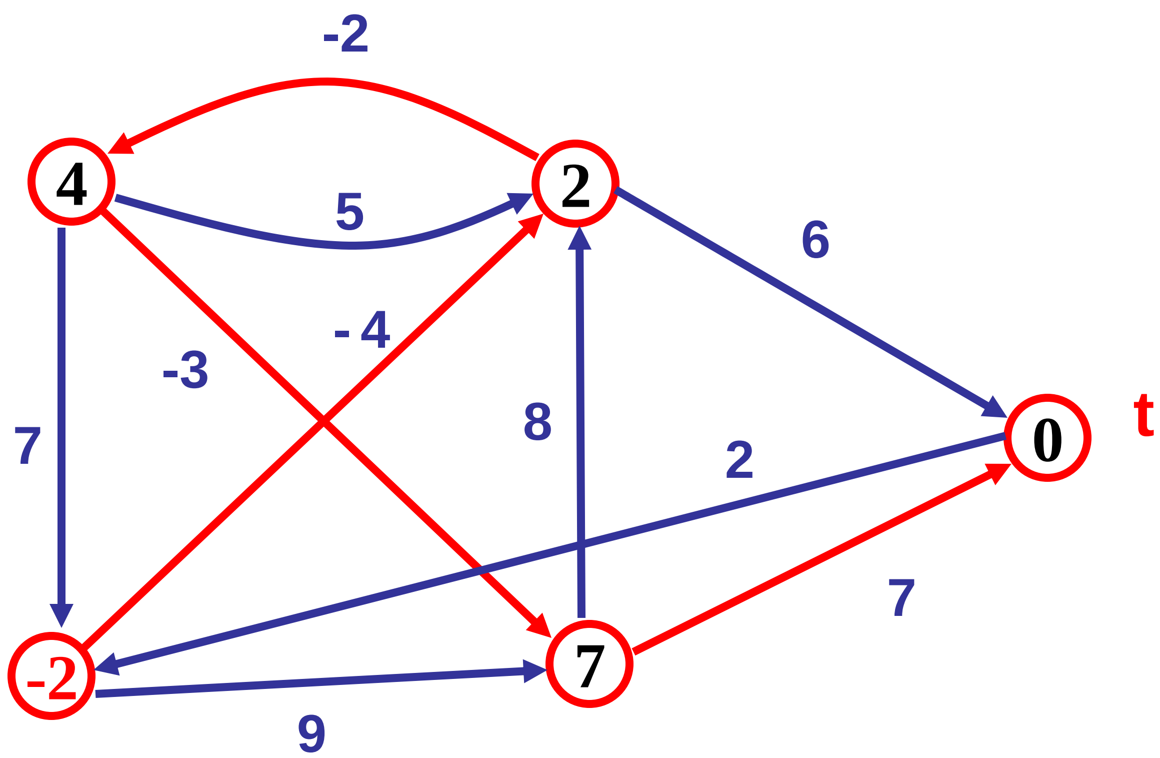
Bellman-Ford



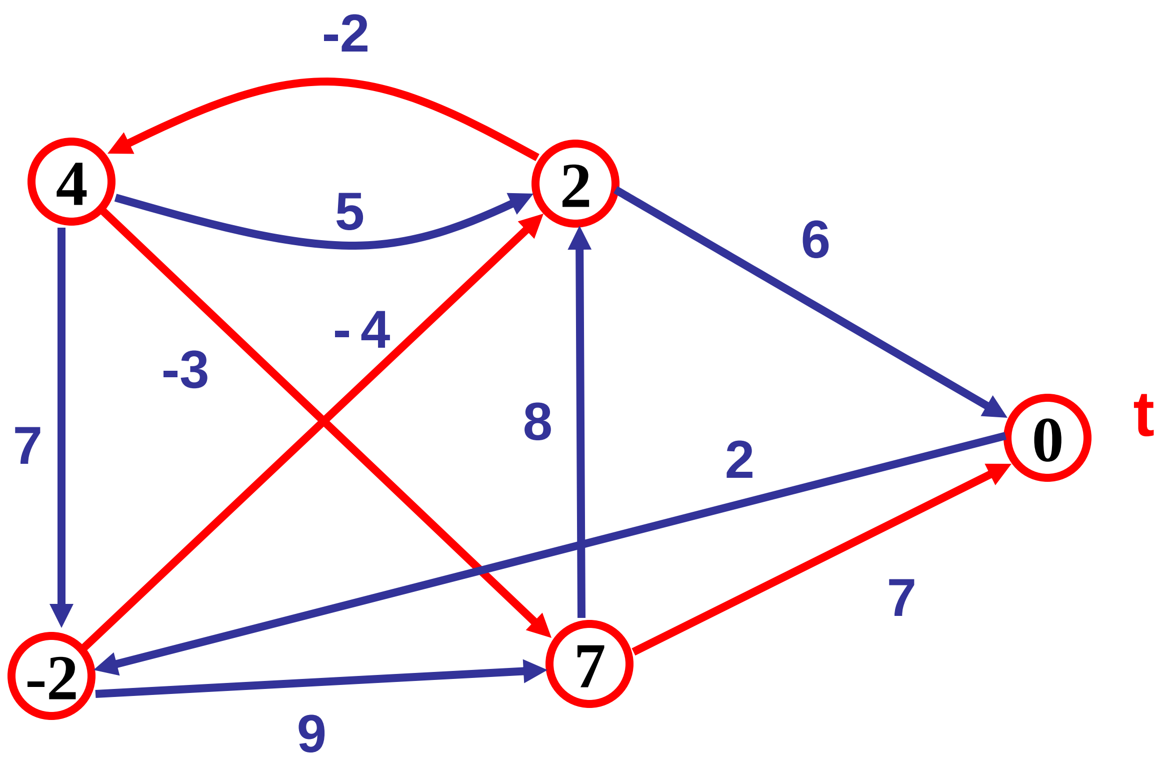
Bellman-Ford



Bellman-Ford



Bellman-Ford



Bellman-Ford with a DAG

Edges only go from lower to higher-numbered vertices

- Update distances in reverse order of topological sort
- Only one pass through vertices required
- O(**n**+**m**) time