CSE 521: Design and Analysis of Algorithms I

Stable Matching

Paul Beame

Matching Residents to Hospitals

- Goal: Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

- Unstable pair: applicant **x** and hospital **y** are unstable if:
    - **x** prefers **y** to their assigned hospital.
    - **y** prefers **x** to one of its admitted students.

- Stable assignment. Assignment with no unstable pairs.
    - Natural and desirable condition.
    - Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

- Goal. Given **n** men and **n** women, find a "suitable" matching.
  - Participants rate members of opposite sex.
  - Each man lists women in order of preference from best to worst.
  - Each woman lists men in order of preference from best to worst.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

Stable Matching Problem

- Perfect matching: everyone is matched monogamously.
  - Each man gets exactly one woman.
  - Each woman gets exactly one man.

- Stability: no incentive for some pair of participants to undermine assignment by joint action.
  - In matching **M**, an unmatched pair **m-w** is unstable if man **m** and woman **w** prefer each other to current partners.
  - Unstable pair **m-w** could each improve by eloping.

- Stable matching: perfect matching with no unstable pairs.

- Stable matching problem. Given the preference lists of **n** men and **n** women, find a stable matching if one exists.

Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?
- A. No. Brenda and Xavier will hook up.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

Stable Matching Problem

- Q. Is assignment X-A, Y-B, Z-C stable?
- A. Yes.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Brenda	Claire
Yuri	Brenda	Amy	Claire
Zoran	Amy	Brenda	Claire

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yuri	Xavier	Zoran
Brenda	Xavier	Yuri	Zoran
Claire	Xavier	Yuri	Zoran

Women's Preference Profile

Stable Roommate Problem

- Q. Do stable matchings always exist?
- A. Not obvious a priori.

- Stable roommate problem.
  - **2n** people; each person ranks others from **1** to **2n-1**.
  - Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
David	A	B	C

A-B, C-D ⇒ B-C unstable
A-C, B-D ⇒ A-B unstable
A-D, B-C ⇒ A-C unstable

- Observation. Stable matchings do not always exist for stable roommate problem.

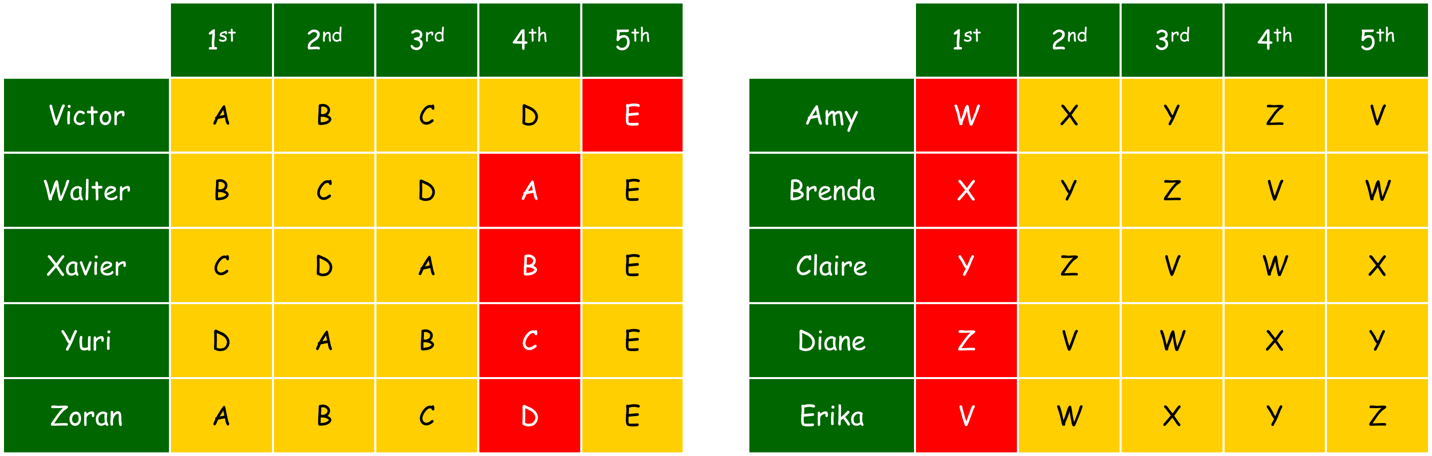
Propose-And-Reject Algorithm

- Propose-and-reject algorithm. [Gale-Shapley 1962]
  Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

- Observation 1. Men propose to women in decreasing order of preference.
- Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."
- Claim. Algorithm terminates after at most n^2 iterations of while loop.
- Proof. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪



	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Walter	B	C	D	A	E
Xavier	C	D	A	B	E
Yuri	D	A	B	C	E
Zoran	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Brenda	X	Y	Z	V	W
Claire	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of Correctness: Perfection

- Claim. All men and women get matched.
- Proof. (by contradiction)
  - Suppose, for sake of contradiction, that Zoran is not matched upon termination of algorithm.
  - Then some woman, say Amy, is not matched upon termination.
  - By Observation 2 (only trading up, never becoming unmatched), Amy was never proposed to.
  - But, Zoran proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

- Claim. No unstable pairs.
- Proof. (by contradiction)
  - Suppose **A-Z** is an unstable pair: each prefers each other to partner in Gale-Shapley matching **S***.
  - Case 1: **Z** never proposed to **A**.

    ⇒ **Z** prefers his GS partner to **A**.

    ⇒ **A-Z** is stable.

men propose in decreasing order of preference

S*
Amy-Yuri
Brenda-Zoran
. . .

  - Case 2: **Z** proposed to **A**.

    ⇒ **A** rejected **Z** (right away or later)

    ⇒ **A** prefers her GS partner to **Z**.

    ⇒ **A-Z** is stable.

women only trade up

  - In either case **A-Z** is stable, a contradiction. ▪

Summary

- Stable matching problem. Given **n** men and **n** women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

- Q. How to implement GS algorithm efficiently?

- Q. If there are multiple stable matchings, which one does GS find?

Implementation for Stable Matching Algorithms

- Problem size
  - **$N=2n^2$** words
    - **2n** people each with a preference list of length **n**
  - **$2n^2$log n** bits
    - specifying an ordering for each preference list takes **n**log **n** bits

- Brute force algorithm
  - Try all **n!** possible matchings
  - Do any of them work?

- Gale-Shapley Algorithm
  - **n^2** iterations, each costing constant time as follows:

Efficient Implementation

- Efficient implementation. We describe $O(n^2)$ time implementation.

- Representing men and women.
  - Assume men are named **1**, …, **n**.
  - Assume women are named **1'**, …, **n'**.

- Engagements.
  - Maintain a list of free men, e.g., in a queue.
  - Maintain two arrays **wife**[**m**], and **husband**[**w**].
    - set entry to **0** if unmatched
    - if **m** matched to **w** then **wife**[**m**]=**w** and **husband**[**w**]=**m**

- Men proposing.
  - For each man, maintain a list of women, ordered by preference.
  - Maintain an array **count**[**m**] that counts the number of proposals made by man **m**.

Efficient Implementation

- Women rejecting/accepting.
  - Does woman **w** prefer man **m** to man **m'**?
  - For each woman, create inverse of preference list of men.
  - Constant time access for each query after **O(n)** preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man **3** to **6**
since `inverse[3]=2` < `7=inverse[6]`

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

	1st	2nd	3rd
Xavier	A	B	C
Yuri	B	A	C
Zoran	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Brenda	X	Y	Z
Claire	X	Y	Z

- An instance with two stable matchings.
  - A-X, B-Y, C-Z.
  - A-Y, B-X, C-Z.

Understanding the Solution

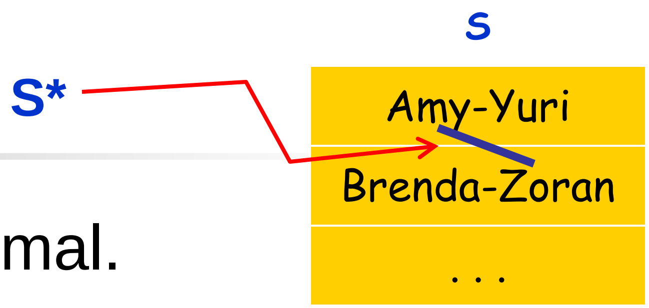
- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

- Def. Man **m** is a valid partner of woman **w** if there exists some stable matching in which they are matched.

- Man-optimal assignment. Each man receives **best** valid partner (according to his preferences).

- Claim. All executions of GS yield a **man-optimal** assignment, which is a stable matching!
  - No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
  - Simultaneously best for each and every man.

Man Optimality



- Claim. GS matching **S*** is man-optimal.
- Proof. (by contradiction)
  - Suppose some man is paired with someone other than his best partner. Men propose in decreasing order of preference ⇒ some man is rejected by a valid partner.
  - Let **Y** be first such man, and let **A** be the first valid woman that rejects him.
  - Let **S** be a stable matching where **A** and **Y** are matched.
  - In building **S***, when **Y** is rejected, **A** forms (or reaffirms) engagement with a man, say **Z**, whom she prefers to **Y**.
  - Let **B** be **Z**'s partner in **S**.
  - In building **S***, **Z** is not rejected by any valid partner at the point when **Y** is rejected by **A**.
  - Thus, **Z** prefers **A** to **B**.
  - But **A** prefers **Z** to **Y**.
  - Thus **A-Z** is unstable in **S**. ▪

since this is the first rejection by a valid partner

Stable Matching Summary

- Stable matching problem. Given preference profiles of **n** men and **n** women, find a stable matching.

no man and woman prefer to be with each other than with their assigned partner

- Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

- Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of **m** if there exist some stable matching where **m** and **w** are paired

- Q. Does man-optimality come at the expense of the women?

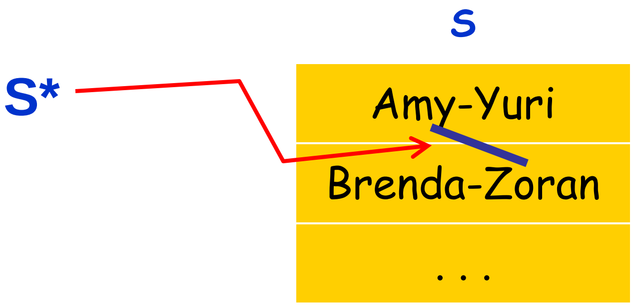
Woman Pessimality

- Woman-pessimal assignment. Each woman receives worst valid partner.

- Claim. GS finds woman-pessimal stable matching **S***.

- Proof.
  - Suppose **A**-**Z** matched in **S***, but **Z** is not worst valid partner for **A**.
  - There exists stable matching **S** in which **A** is paired with a man, say **Y**, whom she likes less than **Z**.
  - Let **B** be **Z**'s partner in **S**.
  - **Z** prefers **A** to **B**.
  - Thus, **A**-**Z** is an unstable in **S**. ▪

man-optimality of S*



Extensions: Matching Residents to Hospitals

- Ex: Men ≈ hospitals, Women ≈ med school residents.
- Variant 1. Some participants declare others as unacceptable.
- Variant 2. Unequal number of men and women.
- Variant 3. Limited polygamy.

e.g. resident **A** unwilling to work in Cleveland

e.g. hospital **X** wants to hire **3** residents

- Def. Matching **S** is unstable if there is a hospital **h** and resident **r** such that:
  - **h** and **r** are acceptable to each other; and
  - either **r** is unmatched, or **r** prefers **h** to her assigned hospital; and
  - either **h** does not have all its places filled, or **h** prefers **r** to at least one of its assigned residents.

Application: Matching Residents to Hospitals

- NRMP. (National Resident Matching Program)
  - Original use just after WWII. ← predates computer usage
  - Ides of March, 23,000+ residents.

- Rural hospital dilemma.
  - Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
  - Rural hospitals were under-subscribed in NRMP matching.
  - How can we find stable matching that benefits "rural hospitals"?

- Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Deceit: Machiavelli Meets Gale-Shapley

- Q. Can there be an incentive to misrepresent your preference profile?
  - Assume you know men's propose-and-reject algorithm will be run.
  - Assume that you know the preference profiles of all other participants.
- Fact. No, for any man. Yes, for some women. No mechanism can guarantee a stable matching and be cheatproof.

	1st	2nd	3rd
Xavier	A	B	C
Yuri	B	A	C
Zoran	A	B	C

Men's Preference List

	1st	2nd	3rd
Amy	Y	X	Z
Brenda	X	Y	Z
Claire	X	Y	Z

Women's True Preference Profile

	1st	2nd	3rd
Amy	Y	Z	X
Brenda	X	Y	Z
Claire	X	Y	Z

Amy Lies