CSE 521: Design and Analysis of Algorithms Assignment #3

due: Tuesday, Oct 25. 10:30AM

Each problem is worth 10 points. KT refers to *Algorithm Design*, First Edition, by Kleinberg and Tardos. "Give an algorithm" means pseudo-code, a high-level explanation and a proof of correctness. See the website for more grading guidelines.

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1. *Convergence of the gradient descent algorithm for solving $Ax = b$:* Consider the following algorithm:

Algorithm 1 Gradient descent algorithm for solving linear systems of equations

$k \leftarrow 0$.
$x_0 \leftarrow 0$.
repeat
  $r_k \leftarrow b - Ax_k$
  $\alpha_k \leftarrow \frac{r_k^T r_k}{r_k^T A r_k}$.
  $x_{k+1} \leftarrow x_k + \alpha_k r_k$.
  $k \leftarrow k + 1$.
until $\|\alpha_k\| < 10^{-20}$

(a) Prove that $r_{k+1}^T r_k = 0$ for all k.

(b) Define $d_k = A^{-1}b - x_k = A^{-1}r_k$. Define $\delta_k = d_k^T A d_k$ to be a measure of error. Prove that

$$\delta_{k+1} \leq (1 - 1/\kappa)\delta_k.$$

Here $\kappa := \|A\| \cdot \|A^{-1}\|$ is the condition number of A, and $\|M\| := \max_{z \neq 0} \frac{z^T M z}{z^T z}$.

2. KT, Chapter 5, Problem 1

3. KT, Chapter 5, Problem 4

4. KT, Chapter 5, Problem 5

5. For two sets X, Y of integers, the Minkowski sum $X + Y$ is the set of all pairwise sums $\{x + y | x \in X, y \in Y\}$. The goal of this problem is to compute $|X + Y|$; that is, the number of elements in $X + Y$. Let $n = |X| = |Y|$ and assume that all elements of X, Y are between 0 and M. Further assume that M is small enough so that adds, multiplies, etc of $O(\log M)$-bit numbers takes constant time.

   (a) Describe an algorithm to compute $|X + Y|$ in time $O(n^2 \log(n))$.

   (b) Describe an algorithm to compute $|X + Y|$ in time $O(M \log(M))$.

   (c) For k a positive integer, define $kX = \overbrace{X + X + \cdots + X}^{k \text{ times}}$. Describe an algorithm to compute $|kX|$ in time $O(kM \log(kM))$.

   (d) *Extra credit:* Let $L = |kX|$. Describe a randomized algorithm to compute $|kX|$ with $\geq 2/3$ probability of success in time $O(L^2 \log(L))$.