# CSE 521
# Algorithms:
# Divide and Conquer

Larry Ruzzo

Thanks to Paul Beame, Kevin Wayne for some slides

# Outline:

- General Idea
- Review of Merge Sort
- Why does it work?
  - Importance of balance
  - Importance of super-linear growth
- Some interesting applications
  - Closest points
  - Integer Multiplication
- Finding & Solving Recurrences

# Divide & Conquer

Reduce problem to one or more sub-problems of the same type

Typically, each sub-problem is at most a constant fraction of the size of the original problem

Subproblems typically disjoint

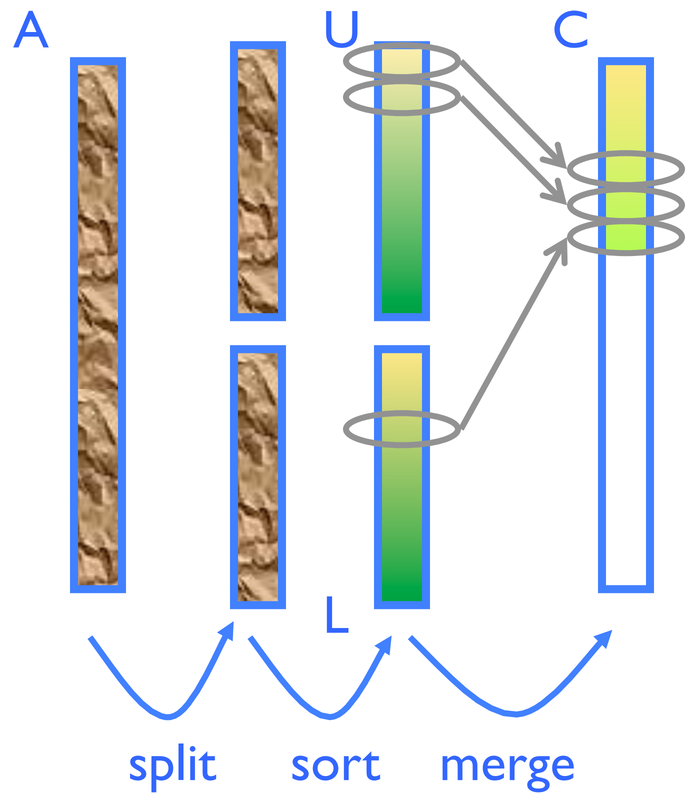
Often gives significant, usually polynomial, speedup

Examples:

- Binary Search, Mergesort, Quicksort (roughly), Strassen's Algorithm, …

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A;
    New U:array[1:n/2] = MS(A[1..n/2]);
    New L:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(U,L));
    }

Merge(U,L: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n
        C[i] = "smaller of U[a], L[b] and correspondingly a++ or b++";
    Return C;
    }
```



Alternative "divide & conquer" algorithm:

Sort n-1

Sort last 1

Merge them

$T(n) = T(n-1)+T(1)+3n \quad \text{for } n \geq 2$

$T(1) = 0$

Solution: $3n + 3(n-1) + 3(n-2) \ldots = \Theta(n^2)$

Suppose we've already invented DumbSort, taking time $n^2$

Try *Just One Level* of divide & conquer:

DumbSort(first n/2 elements)

DumbSort(last n/2 elements)

Merge results

Time: $\boxed{2\,(n/2)^2 + n = n^2/2 + n \ll n^2}$

D&C in a nutshell

*Almost twice as fast!*

Moral 1: “two halves are better than a whole”

Two problems of half size are *better* than one full-size problem, even given O(n) overhead of recombining, since the base algorithm has *super-linear* complexity.

Moral 2: “If a little's good, then more's better”

Two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing. Best is usually full recursion down to some small constant size (balancing "work" vs "overhead").

In the limit: you’ve just rediscovered mergesort!

# Moral 3: unbalanced division less good:

$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$

The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can't get 50-50 split, but balanced is better if you can.

This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

$(1)^2 + (n-1)^2 + n = n^2 - 2n + 2 + n$

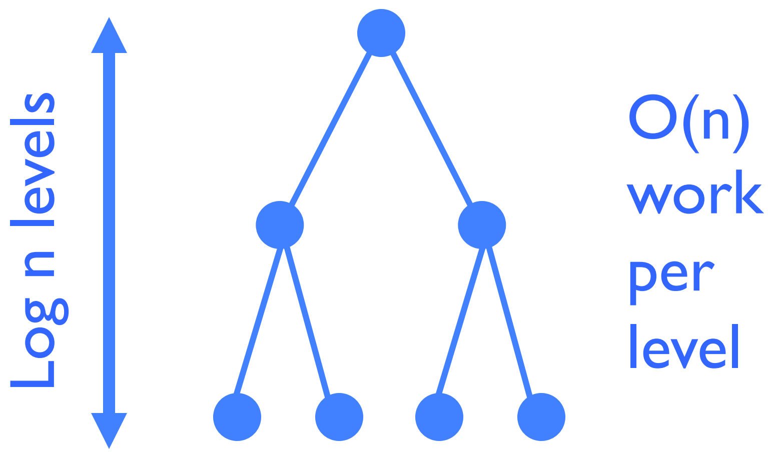
Little improvement here.

Mergesort: (recursively) sort 2 half-lists, then merge results.

$T(n) = 2T(n/2)+cn, \quad n \geq 2$

$T(1) = 0$

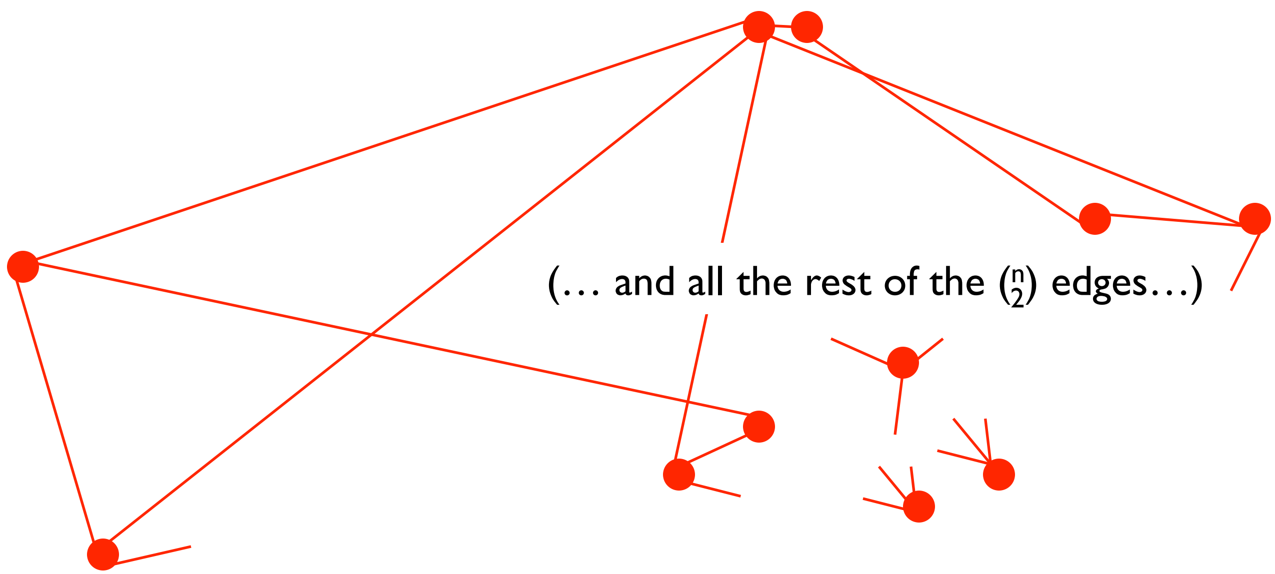
Solution: $\Theta(n \log n)$
(details later)



# A Divide & Conquer Example: Closest Pair of Points

# closest pair of points: non-geometric version

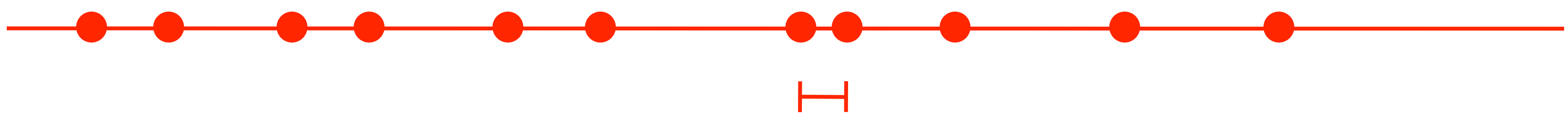
Given n points and *arbitrary* distances between them, find the closest pair. (E.g., think of distance as airfare – definitely *not* Euclidean distance!)




*Must* look at all n choose 2 pairwise distances, else any one you didn't check might be the shortest.

Also true for Euclidean distance in 1-2 dimensions?

Given n points on the real line, find the closest pair



Closest pair is *adjacent* in ordered list

Time O(n log n) to sort, if needed

Plus O(n) to scan adjacent pairs

Key point: do *not* need to calc distances between all pairs: exploit geometry + ordering

# closest pair of points: 2 dimensional version

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

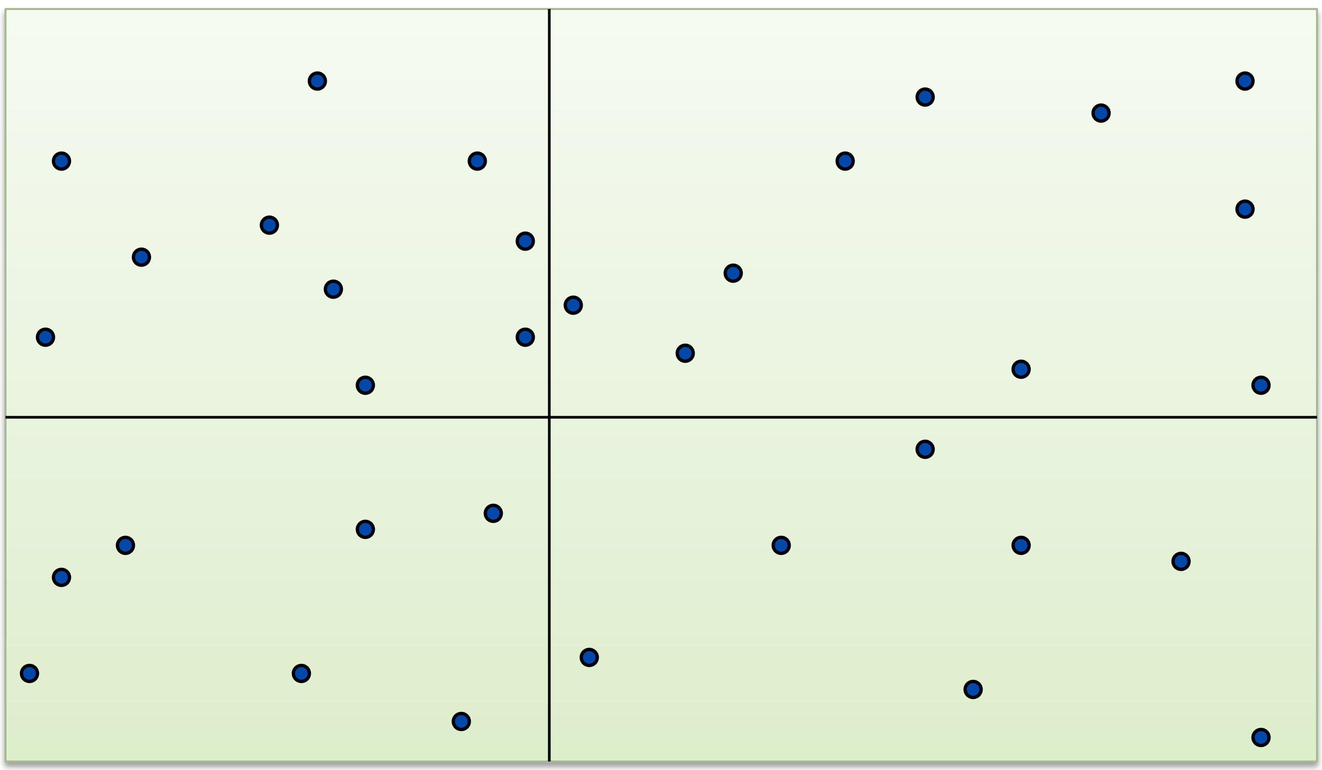
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

Just to simplify presentation

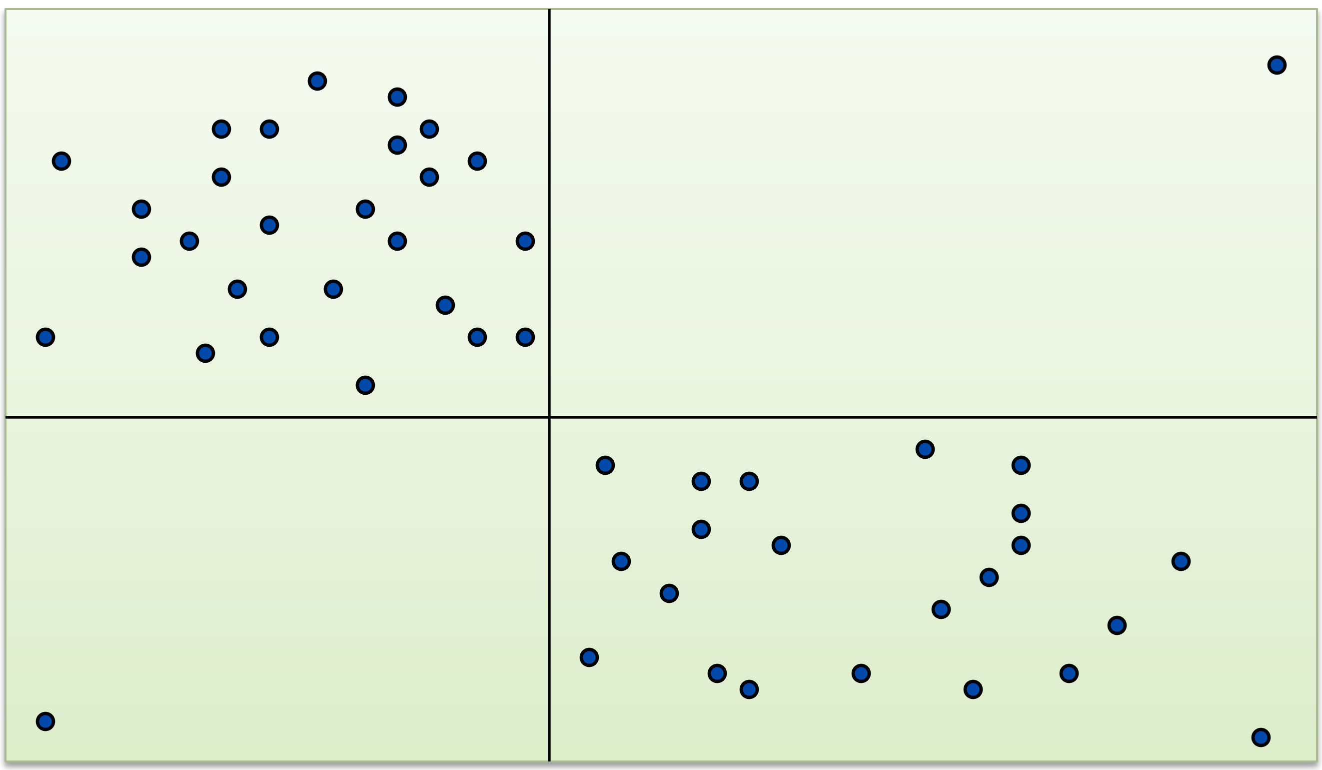
# closest pair of points. 2d, Euclidean distance: 1st try

Divide. Sub-divide region into 4 quadrants.



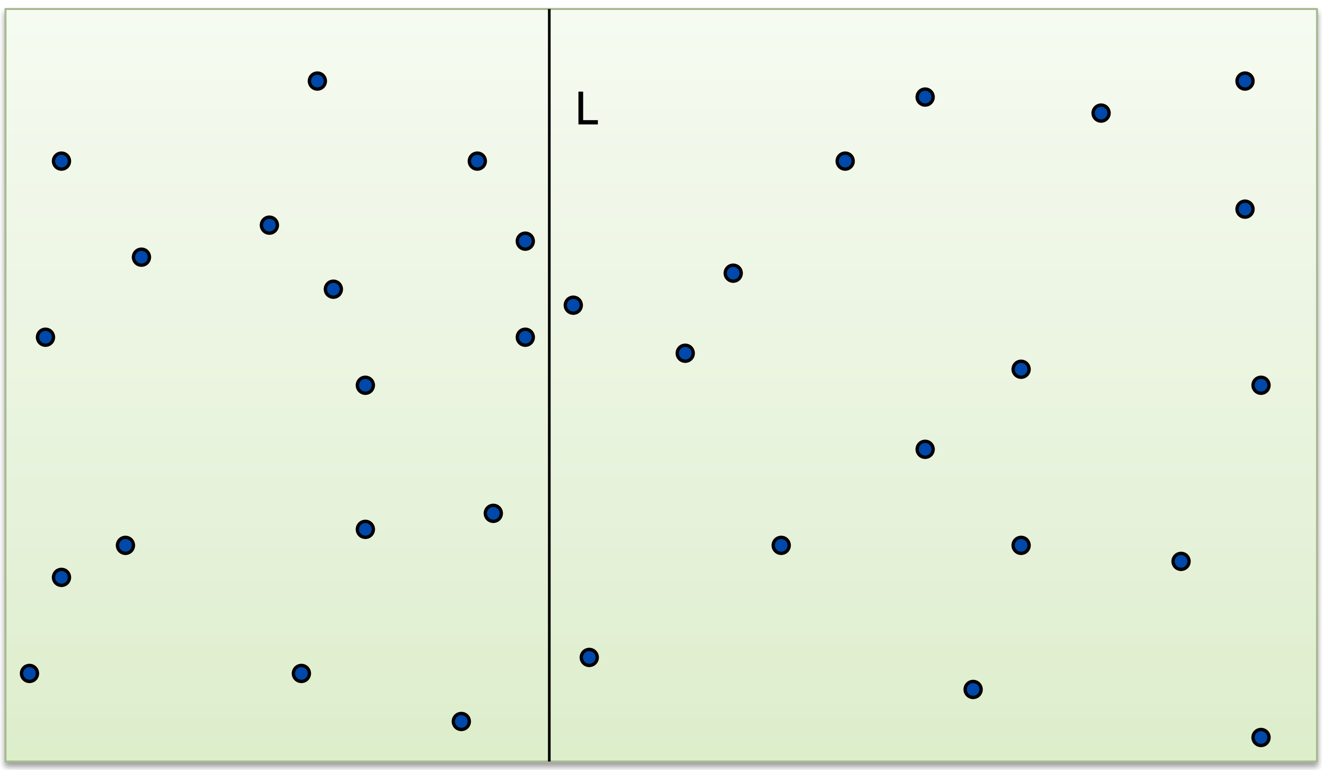
Divide. Sub-divide region into 4 quadrants.

Obstacle. Impossible to ensure n/4 points in each piece.



# Algorithm.

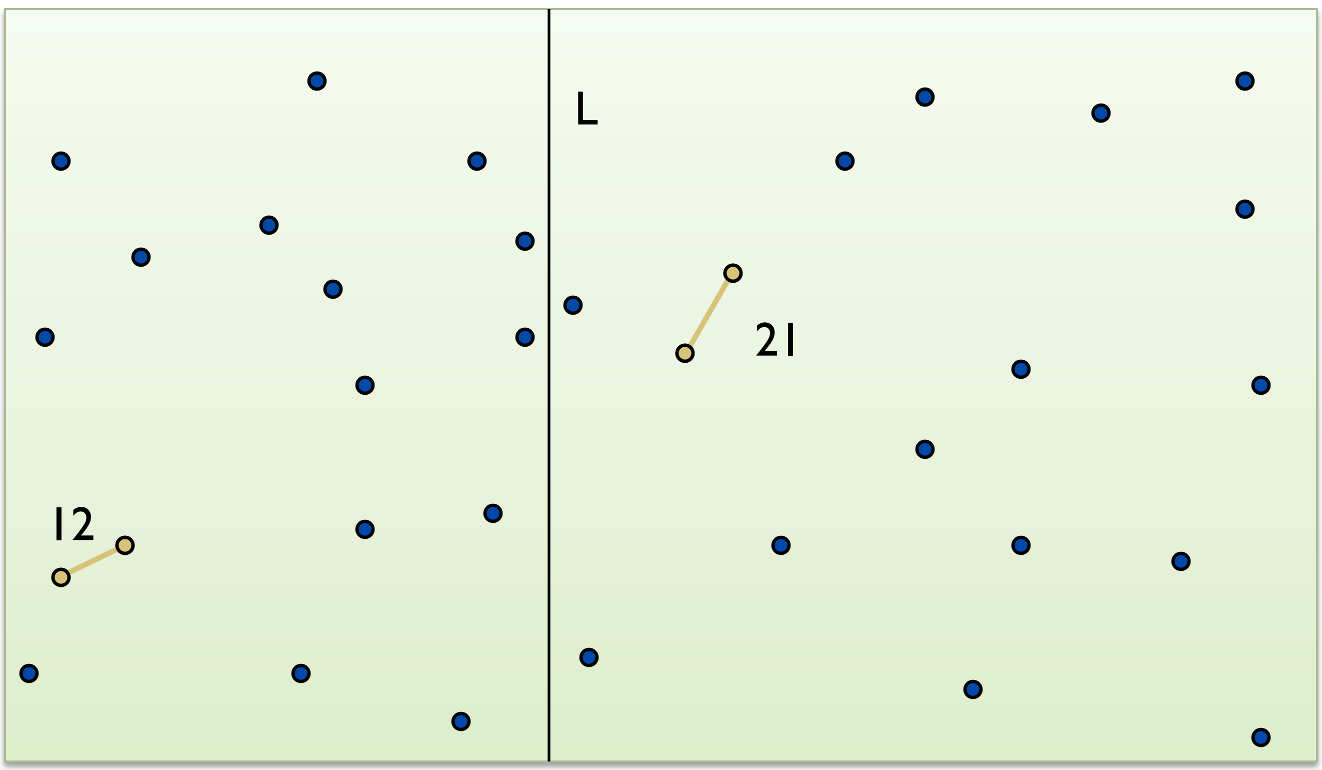
Divide: draw vertical line L with ≈ n/2 points on each side.



# Algorithm.

Divide: draw vertical line L with ≈ n/2 points on each side.

Conquer: find closest pair on each side, recursively.



# Algorithm.

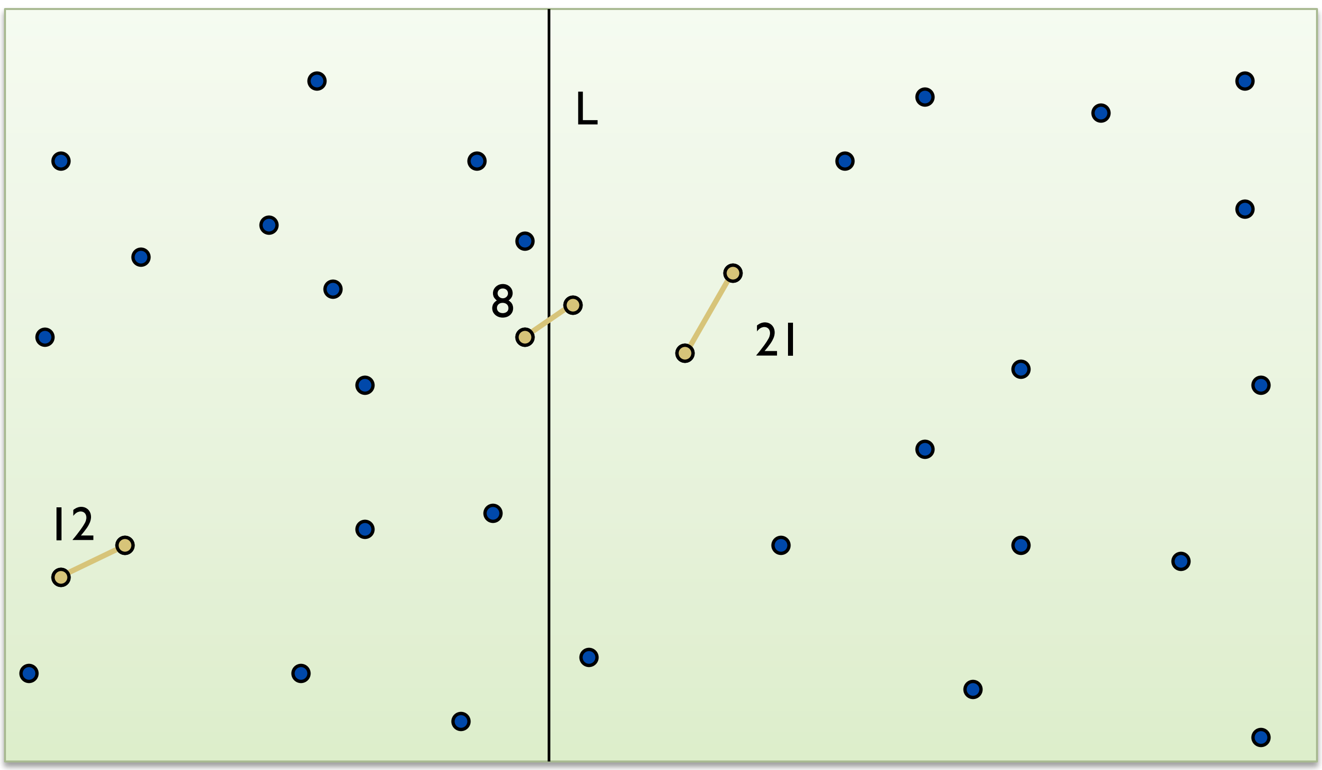
Divide: draw vertical line L with ≈ n/2 points on each side.

Conquer: find closest pair on each side, recursively.

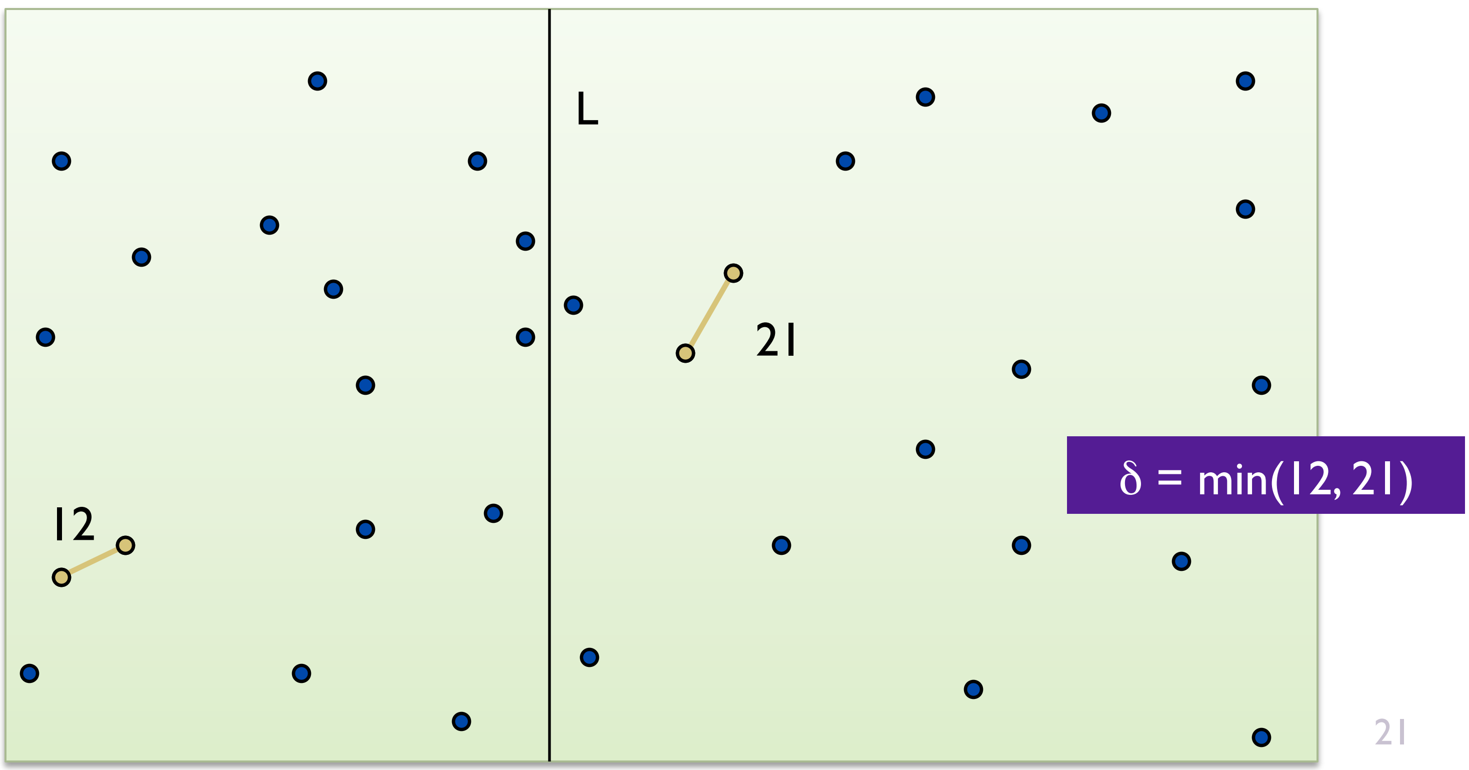
Combine: find closest pair with one point in each side.

Return best of 3 solutions.

seems like $\Theta(n^2)$ ?

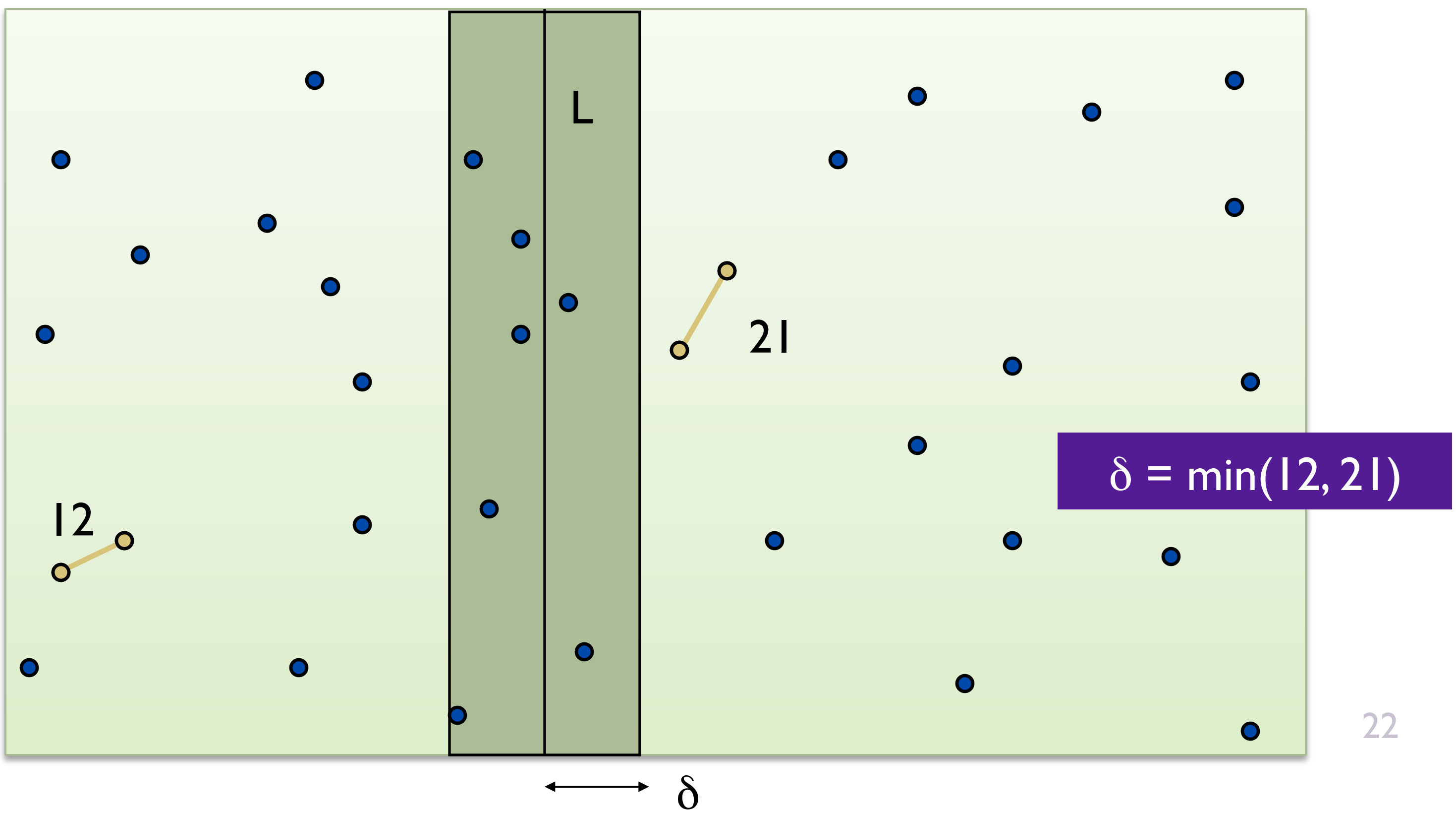


Find closest pair with one point in each side, *assuming* distance < δ.



Find closest pair with one point in each side, *assuming distance < δ.*

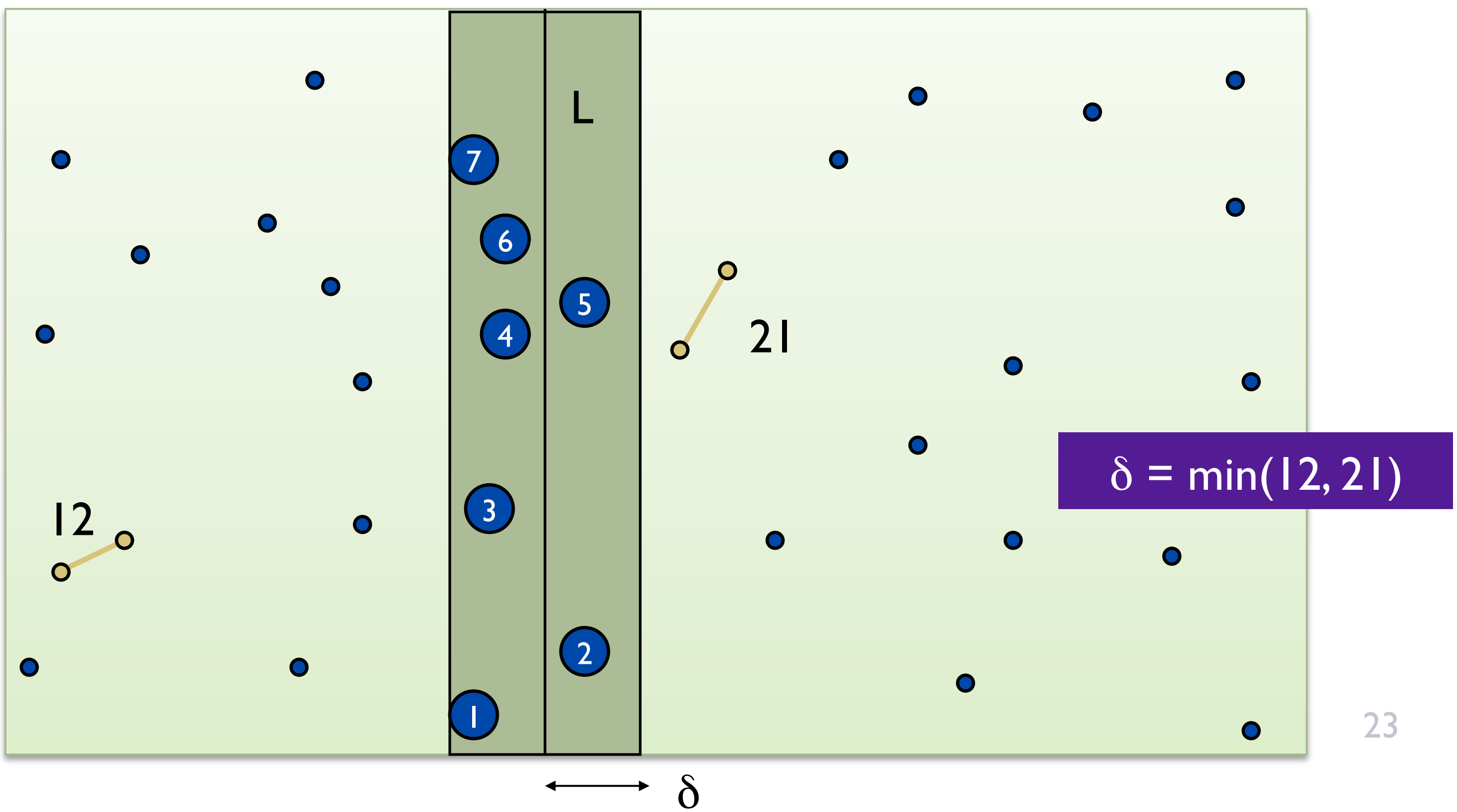
Observation: suffices to consider points within δ of line L.



# Find closest pair with one point in each side, *assuming* distance < δ.

Observation: suffices to consider points within δ of line L.

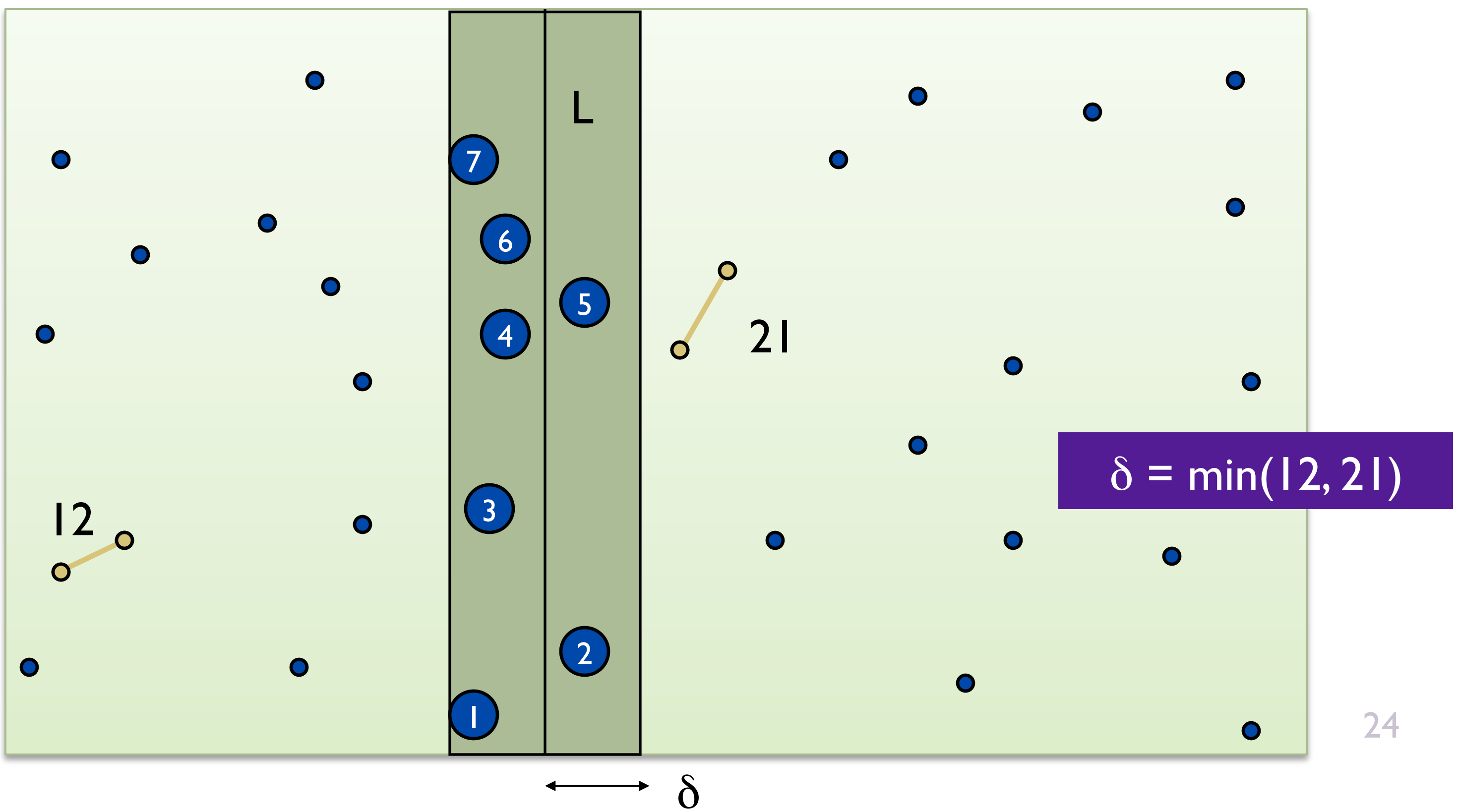
Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate.



# Find closest pair with one point in each side, *assuming* distance < δ.

Observation: suffices to consider points within δ of line L.

Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate. Only check pts within 8 in sorted list!



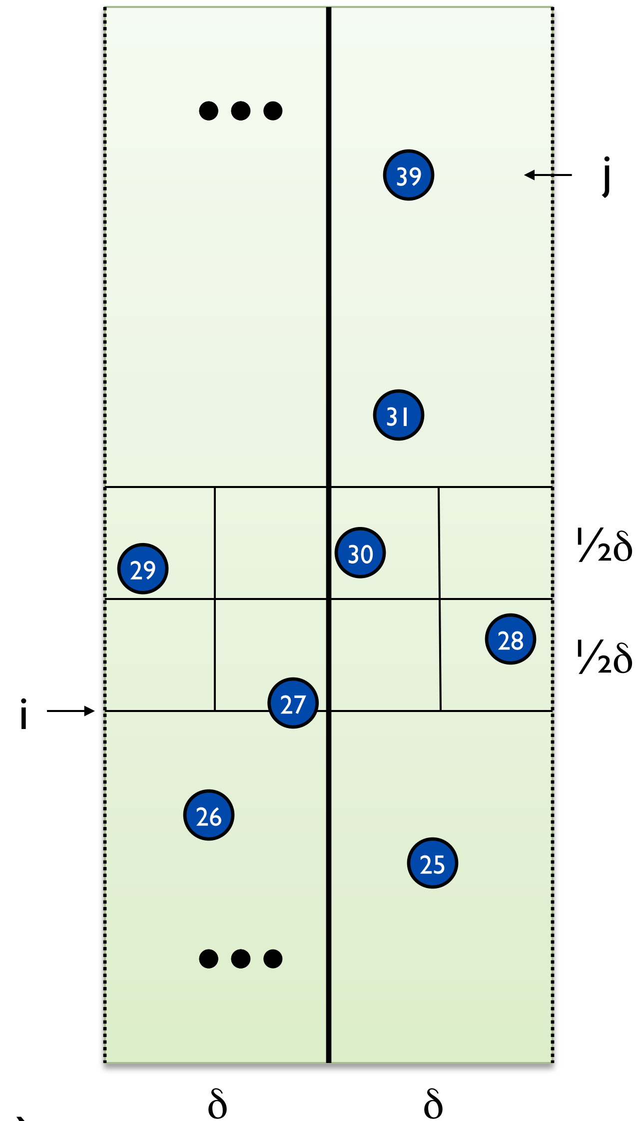
Def. Let $s_i$ have the $i^{th}$ smallest y-coordinate among points in the 2δ-width-strip.

Claim. If $|i - j| > 8$, then the distance between $s_i$ and $s_j$ is $> \delta$.

Pf: No two points lie in the same ½δ-by-½δ box:

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \approx 0.7 < 1$$

so ≤ 8 boxes within +δ of $y(s_i)$.



```
Closest-Pair(p₁, …, pₙ) {
   if(n <= ??) return ??

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```

Analysis, I: Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points

$$D(n) \le \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT – that's only the number of *distance calculations*

What if we counted comparisons?

Analysis, II: Let C(n) be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points

$$C(n) \le \left\{ \begin{array}{ccc} 0 & & n=1 \\ 2C\left(n/2\right) + kn\log n & & n>1 \end{array} \right\} \Rightarrow C(n) = O(n \log^2 n)$$

for some constant $k$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points from scratch each time.

Sort by x at top level only.

Each recursive call returns δ *and* list of all points sorted by y

Sort by merging two pre-sorted lists.

$$T(n) \le 2T\left(n/2\right) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

Code is longer & more complex

O(n log n) vs O($n^2$) may hide 10x in constant?

How many points?

| n | Speedup: $n^2$ / (10 n $\log_2$ n) |
|---|---|
| 10 | 0.3 |
| 100 | 1.5 |
| 1,000 | 10 |
| 10,000 | 75 |
| 100,000 | 602 |
| 1,000,000 | 5,017 |
| 10,000,000 | 43,004 |

# Going From Code to Recurrence

Carefully define what you're counting, and *write it down*!

"Let C(n) be the number of comparisons between sort keys used by MergeSort when sorting a list of length n ≥ 1"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted.*

Write Recurrence(s)

Base Case

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A;
    New L:array[1:n/2] = MS(A[1..n/2]);
    New R:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(L,R));
    }
Merge(A,B: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n {
        C[i] = "smaller of A[a], B[b] and a++ or b++";
    Return C;
    }
```

Recursive calls

One Recursive Level

Operations being counted

Base case

$$C(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2C(n/2) + (n-1) & \text{if } n > 1 \end{cases}$$

Recursive calls

One compare per element added to merged list, except the last.

Total time: proportional to C(n)

(loops, copying data, parameter passing, etc.)

Carefully define what you're counting, and *write it down*!

"Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on $n \geq 1$ points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

Basic operations: distance calcs

# closest pair algorithm

```
Closest-Pair(p₁, …, pₙ) {
   if(n <= 1) return ∞

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
         δ = min(δ, distance between p[i] and p[i+k]);
         k++;

   return δ.
}
```

Base Case — 0

Recursive calls (2) — 2D(n / 2)

Basic operations at this recursive level — 7n

One recursive level

Analysis, I: Let D(n) be the number of pairwise distance calculations in the Closest-Pair Algorithm when run on n ≥ 1 points

$$D(n) \le \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT – that's only the number of *distance calculations*

What if we counted comparisons?

Carefully define what you're counting, and *write it down*!

> "Let D(n) be the number of comparisons between coordinates/distances in the Closest-Pair Algorithm when run on $n \geq 1$ points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

# closest pair algorithm

Basic operations:
*comparisons*

Recursive calls (2)

Base Case

```
Closest-Pair(p₁, …, pₙ) {
   if(n <= 1) return ∞

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
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   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```

0

$k_1 n \log n$

$2C(n / 2)$

1

$k_2 n$

$k_3 n \log n$

Basic operations at this recursive level

$7n$

One recursive level

Analysis, II: Let C(n) be the number of comparisons of coordinates/distances in the Closest-Pair Algorithm when run on n ≥ 1 points

$$C(n) \le \begin{cases} 0 & n = 1 \\ 2C(n/2) + k_4 n \log n & n > 1 \end{cases} \Rightarrow C(n) = O(n \log^2 n)$$

for some $k_4 \le k_1 + k_2 + k_3 + 7$

Q. Can we achieve time O(n log n)?

A. Yes. Don't sort points from scratch each time.

Sort by x at top level only.

Each recursive call returns δ *and* list of all points sorted by y

Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

# Integer Multiplication

Add. Given two n-bit integers a and b, compute a + b.

Add

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| + | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |

O(n) bit operations.

Add. Given two n-bit integers a and b, compute a + b.

$\boxed{O(n) \text{ bit operations.}}$

Add

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| + | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |

Multiply. Given two n-bit integers a and b, compute a × b.
The "grade school" method:

$\boxed{\Theta(n^2) \text{ bit operations.}}$

Multiply

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | | | | | | | * | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| | | | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| | | | | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | |
| | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | |
| | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | | |
| | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | | | |
| | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | | | | | | |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

# To multiply two 2-digit integers:

Multiply four 1-digit integers.

Add, shift some 2-digit integers to obtain result.

$$
\begin{aligned}
x &= 10 \cdot x_1 + x_0 \\
y &= 10 \cdot y_1 + y_0 \\
xy &= (10 \cdot x_1 + x_0)(10 \cdot y_1 + y_0) \\
&= 100 \cdot x_1 y_1 + 10 \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0
\end{aligned}
$$

Same idea works for *long* integers –

can split them into 4 half-sized ints

| | | | | |
|---|---|---|---|---|
| | | 4 | 5 | $y_1$ $y_0$ |
| | | 3 | 2 | $x_1$ $x_0$ |
| | | 1 | 0 | $x_0 \cdot y_0$ |
| | 0 | 8 | | $x_0 \cdot y_1$ |
| | 1 | 5 | | $x_1 \cdot y_0$ |
| 1 | 2 | | | $x_1 \cdot y_1$ |
| 1 | 4 | 4 | 0 | |

# To multiply two n-bit integers:

Multiply four ½n-bit integers.

Add two ½n-bit integers, and shift to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

$$
\mathrm{T}(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) = \Theta(n^2)
$$

assumes n is a power of 2

| | | | | |
|---|---|---|---|---|
| | | 1 1 0 1 | 0 1 0 1 | $y_1$ $y_0$ |
| | | * 0 1 1 1 | 1 1 0 1 | $x_1$ $x_0$ |
| | | 0 1 0 0 | 0 0 0 1 | $x_0 \cdot y_0$ |
| | 1 0 1 0 | 1 0 0 1 | | $x_0 \cdot y_1$ |
| | 0 0 1 0 | 0 0 1 1 | | $x_1 \cdot y_0$ |
| 0 1 0 1 | 1 0 1 1 | | | $x_1 \cdot y_1$ |
| 0 1 1 0 | 1 0 0 0 | 0 0 0 0 | 0 0 0 1 | |

# key trick: 2 multiplies for the price of 1:

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

Well, ok, 4 for 3 is more accurate…

$$
\begin{aligned}
\alpha &= x_1 + x_0 \\
\beta &= y_1 + y_0 \\
\alpha\beta &= \left(x_1 + x_0\right)\left(y_1 + y_0\right) \\
&= x_1 y_1 + \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
\left(x_1 y_0 + x_0 y_1\right) &= \alpha\beta - x_1 y_1 - x_0 y_0
\end{aligned}
$$

To multiply two n-bit integers:

- Add two ½n bit integers.
- Multiply three ½n-bit integers.
- Add, subtract, and shift ½n-bit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2}\cdot x_1 + x_0 \\
y &= 2^{n/2}\cdot y_1 + y_0 \\
xy &= 2^{n}\cdot x_1 y_1 + 2^{n/2}\cdot\left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
&= 2^{n}\cdot \underset{A}{x_1 y_1} + 2^{n/2}\cdot\left(\underset{B}{(x_1 + x_0)(y_1 + y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0}\right) + \underset{C}{x_0 y_0}
\end{aligned}
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$
\mathrm{T}(n) \le \underbrace{T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + T\left(1 + \lceil n/2 \rceil\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}
$$

$$
\textit{Sloppy version}:\ T(n) \le 3T(n/2) + O(n)
$$

$$
\Rightarrow\ \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$\mathrm{T}(n) \le \underbrace{T\big(\lfloor n/2 \rfloor\big) + T\big(\lceil n/2 \rceil\big) + T\big(1+\lceil n/2 \rceil\big)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}$$

$$\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)$$

$$\Rightarrow \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$T(n) \le \underbrace{T\big(\lfloor n/2 \rfloor\big) + T\big(\lceil n/2 \rceil\big) + T\big(1 + \lceil n/2 \rceil\big)} + \underbrace{\Theta(n)}$$

$$\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)$$

$$\Rightarrow T(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

$n \to 2^{\lceil \log_2 n \rceil}$

## Karatsuba multiplication

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$T(n) \le \underbrace{T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + T\left(1 + \lceil n/2 \rceil\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}$$

$$\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)$$

$$\Rightarrow \; T(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

$T(n/2)$

$x \; 1$

$y \; 0$

$00$

$+ O(n)$

$xy$

Naïve: $\Theta(n^2)$

Karatsuba: $\Theta(n^{1.59\ldots})$

Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems $\rightarrow \Theta(n^{1.46\ldots})$

Best known: $\Theta(n \log n \log\log n)$

"Fast Fourier Transform"

but mostly unused in practice (unless you need really big numbers - a billion digits of $\pi$, say)

High precision arithmetic *IS* important for crypto

# Another D&C Example: Multiplying Polynomials

Similar ideas apply to polynomial multiplication

We'll describe the basic ideas by multiplying polynomials rather than integers

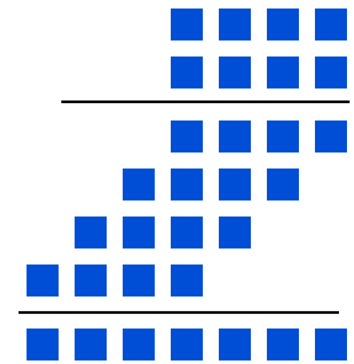
In fact, it's somewhat simpler: no carries!

# Notes on Polynomials

These are just formal sequences of coefficients so when we show something multiplied by $x^k$ it just means shifted $k$ places to the left – basically no work

Usual Polynomial Multiplication:

$$
\begin{array}{r}
3x^2 + 2x + 2 \\
x^2 - 3x + 1 \\
\hline
3x^2 + 2x + 2 \\
-9x^3 - 6x^2 - 6x \phantom{{}+2} \\
3x^4 + 2x^3 + 2x^2 \phantom{{}-6x+2} \\
\hline
3x^4 - 7x^3 - x^2 - 4x + 2
\end{array}
$$



Given:

Degree $m-1$ polynomials $P$ and $Q$

$$P = a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-2}x^{m-2} + a_{m-1}x^{m-1}$$

$$Q = b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-2}x^{m-2} + b_{m-1}x^{m-1}$$

Compute:

Degree $2m-2$ Polynomial $P Q$

$$P Q = a_0 b_0 + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, x^2 + \ldots + (a_{m-2} b_{m-1} + a_{m-1} b_{m-2})\, x^{2m-3} + a_{m-1} b_{m-1}\, x^{2m-2}$$

Obvious Algorithm:

Compute all $a_i b_j$ and collect terms

**$\Theta(m^2)$** time

# Naïve Divide and Conquer

Assume $m=2k$

$$P = (a_0 + a_1 \; x + a_2 \, x^2 + \dots + a_{k-2} \, x^{k-2} + a_{k-1} \, x^{k-1}) +$$
$$(a_k + a_{k+1} \, x + \qquad \dots + a_{m-2} x^{k-2} + a_{m-1} x^{k-1}) \, x^k$$
$$= P_0 + P_1 \, x^k$$

$$Q = Q_0 + Q_1 \, x^k$$

$$PQ = (P_0 + P_1 x^k)(Q_0 + Q_1 x^k)$$
$$= P_0 Q_0 + (P_1 Q_0 + P_0 Q_1) x^k + P_1 Q_1 x^{2k}$$

4 sub-problems of size $k=m/2$ plus linear combining

$T(m)=4T(m/2)+cm$

Solution $T(m) = O(m^2)$

# Karatsuba's Algorithm

A better way to compute terms

Compute

$P_0Q_0$

$P_1Q_1$

$(P_0+P_1)(Q_0+Q_1)$ which is $P_0Q_0+\underline{P_1Q_0+P_0Q_1}+P_1Q_1$

Then

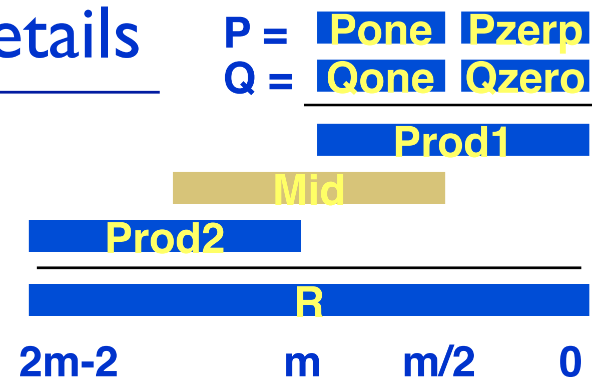
$P_0Q_1+P_1Q_0 = (P_0+P_1)(Q_0+Q_1) - P_0Q_0 - P_1Q_1$

3 sub-problems of size m/2 plus O(m) work

$T(m) = 3\ T(m/2) + cm$

$T(m) = O(m^{\alpha})$ where $\alpha = \log_2 3 = 1.585...$

# Karatsuba: Details




PolyMul(P, Q):

```
// P, Q are length m = 2k vectors, with P[i], Q[i] being
// the coefficient of x^i in polynomials P, Q respectively.
if (m==1) return (P[0]*Q[0]);
Let Pzero be elements 0..k-1 of P; Pone be elements k..m-1
Qzero, Qone : similar
Prod1 = PolyMul(Pzero, Qzero);         // result is a (2k-1)-vector
Prod2 = PolyMul(Pone, Qone);           // ditto
Pzo = Pzero + Pone;                    // add corresponding elements
Qzo = Qzero + Qone;                    // ditto
Prod3 = PolyMul(Pzo, Qzo);             // another (2k-1)-vector
Mid = Prod3 – Prod1 – Prod2;           // subtract corr. elements
R = Prod1 + Shift(Mid, m/2) + Shift(Prod2,m) // a (2m-1)-vector
Return( R );
```

# Multiplication – The Bottom Line

## Polynomials

Naïve: $\Theta(n^2)$

Karatsuba: $\Theta(n^{1.585\ldots})$

Best known: $\Theta(n \log n)$

"Fast Fourier Transform"

## Integers

Similar, but some ugly details re: carries, etc.
gives $\Theta(n \log n \log\log n)$,

but mostly unused in practice

Idea:

"Two halves are better than a whole"

if the base algorithm has super-linear complexity.

"If a little's good, then more's better"

repeat above, recursively

Applications: Many.

Binary Search, Merge Sort, (Quicksort), Closest points, Integer multiply,…

## Recurrences

Above: Where they come from, how to find them

Next: how to solve them

Mergesort: (recursively) sort 2 half-lists, then merge results.

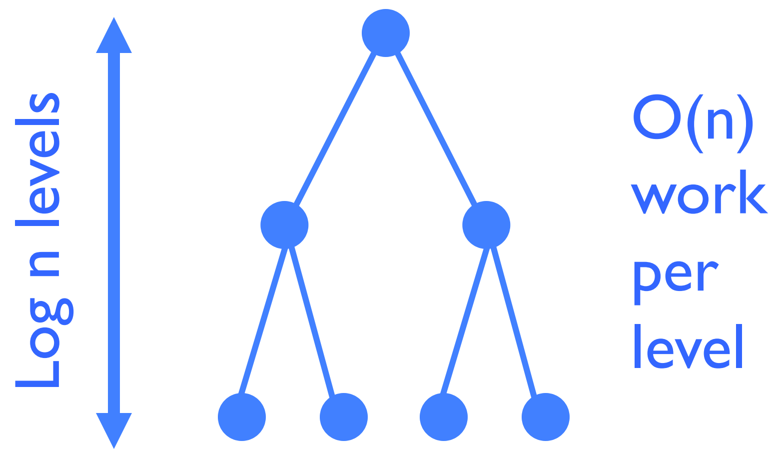
$T(n) = 2T(n/2)+cn, \quad n \geq 2$

$T(1) = 0$

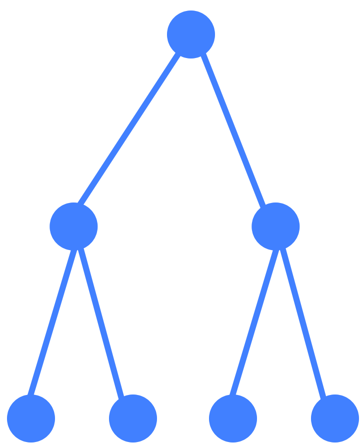
Solution: $\Theta(n \log n)$

(~~details~~ later)

**now**



Solve: $T(1) = c$
$T(n) = 2\,T(n/2) + cn$



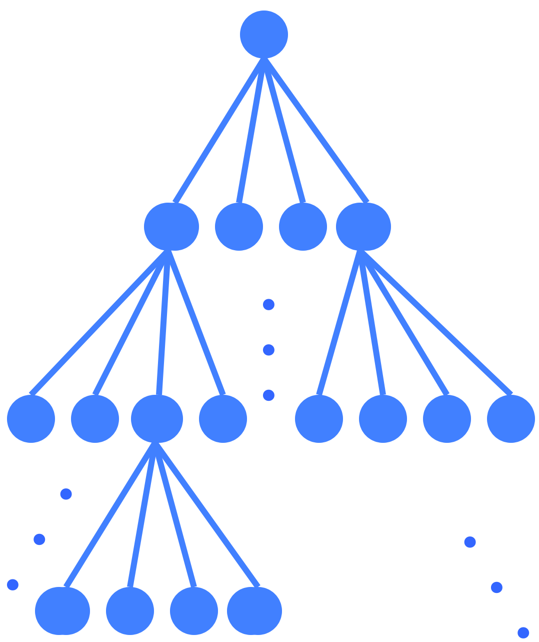
| Level | Num | Size | Work |
| --- | --- | --- | --- |
| 0 | $1 = 2^0$ | $n$ | $cn$ |
| 1 | $2 = 2^1$ | $n/2$ | $2cn/2$ |
| 2 | $4 = 2^2$ | $n/4$ | $4cn/4$ |
| … | … | … | … |
| $i$ | $2^i$ | $n/2^i$ | $2^i\ c\ n/2^i$ |
| … | … | … | … |
| $k-1$ | $2^{k-1}$ | $n/2^{k-1}$ | $2^{k-1}\ c\ n/2^{k-1}$ |
| $k$ | $2^k$ | $n/2^k = 1$ | $2^k\,T(1)$ |

$n = 2^k$ ; $k = \log_2 n$

Total Work: $c\ n\ (1+\log_2 n)$ (add last col)

Solve: $T(1) = c$

$T(n) = 4\,T(n/2) + cn$



| Level | Num | Size | Work |
|---|---|---|---|
| 0 | $1 = 4^0$ | $n$ | $cn$ |
| 1 | $4 = 4^1$ | $n/2$ | $4cn/2$ |
| 2 | $16 = 4^2$ | $n/4$ | $16cn/4$ |
| … | … | … | … |
| $i$ | $4^i$ | $n/2^i$ | $4^i\ c\ n/2^i$ |
| … | … | … | … |
| $k-1$ | $4^{k-1}$ | $n/2^{k-1}$ | $4^{k-1}\ c\ n/2^{k-1}$ |
| $k$ | $4^k$ | $n/2^k = 1$ | $4^k\,T(1)$ |

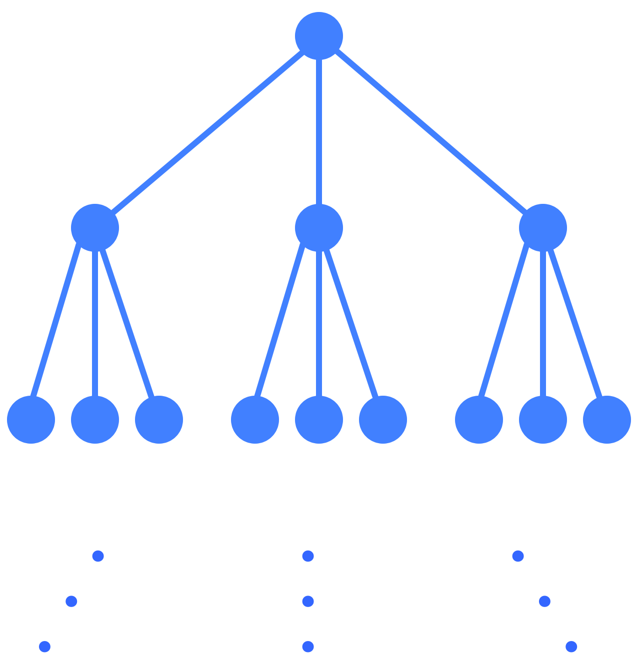
$n = 2^k$ ; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 4^i cn/2^i = O(n^2)$

$4^k = (2^2)^k = (2^k)^2 = n^2$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$



| Level | Num | Size | Work |
|---|---|---|---|
| 0 | $1 = 3^0$ | $n$ | $cn$ |
| 1 | $3 = 3^1$ | $n/2$ | $3cn/2$ |
| 2 | $9 = 3^2$ | $n/4$ | $9cn/4$ |
| … | … | … | … |
| $i$ | $3^i$ | $n/2^i$ | $3^i\ c\ n/2^i$ |
| … | … | … | … |
| $k-1$ | $3^{k-1}$ | $n/2^{k-1}$ | $3^{k-1}\ c\ n/2^{k-1}$ |
| $k$ | $3^k$ | $n/2^k = 1$ | $3^k\,T(1)$ |

$n = 2^k$ ; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$

Theorem:

$$1 + x + x^2 + x^3 + \ldots + x^k = (x^{k+1}-1)/(x-1)$$

proof:

$$
\begin{aligned}
y &= 1 + x + x^2 + x^3 + \ldots + x^k \\
xy &= \phantom{1 + {}} x + x^2 + x^3 + \ldots + x^k + x^{k+1} \\
xy - y &= x^{k+1} - 1 \\
y(x-1) &= x^{k+1} - 1 \\
y &= (x^{k+1}-1)/(x-1)
\end{aligned}
$$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$ (cont.)

$$T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$$

$$= cn \sum_{i=0}^{k} 3^i / 2^i$$

$$= cn \sum_{i=0}^{k} \left(\tfrac{3}{2}\right)^i$$

$$= cn \frac{\left(\tfrac{3}{2}\right)^{k+1} - 1}{\left(\tfrac{3}{2}\right) - 1}$$

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1} \quad (x \neq 1)$$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$ (cont.)

$$cn\frac{\left(\frac{3}{2}\right)^{k+1}-1}{\left(\frac{3}{2}\right)-1} = 2cn\left(\left(\frac{3}{2}\right)^{k+1}-1\right)$$

$$< 2cn\left(\frac{3}{2}\right)^{k+1}$$

$$= 3cn\left(\frac{3}{2}\right)^{k}$$

$$= 3cn\frac{3^k}{2^k}$$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$ (cont.)

$$3cn\frac{3^k}{2^k} = 3cn\frac{3^{\log_2 n}}{2^{\log_2 n}}$$

$$= 3cn\frac{3^{\log_2 n}}{n}$$

$$= 3c3^{\log_2 n}$$

$$= 3c\left(n^{\log_2 3}\right)$$

$$\boxed{= O\left(n^{1.585\ldots}\right)}$$

$$a^{\log_b n}$$

$$= \left(b^{\log_b a}\right)^{\log_b n}$$

$$= \left(b^{\log_b n}\right)^{\log_b a}$$

$$= n^{\log_b a}$$

$T(n) = aT(n/b)+cn^k$ for $n > b$ then

$a > b^k \Rightarrow T(n) = \Theta(n^{\log_b a})$ [many subprobs → leaves dominate]

$a < b^k \Rightarrow T(n) = \Theta(n^k)$ [few subprobs → top level dominates]

$a = b^k \Rightarrow T(n) = \Theta(n^k \log n)$ [balanced → all log n levels contribute]

Fine print:

$a \geq 1$; $b > 1$; $c, d, k \geq 0$; $T(1) = d$; $n = b^t$ for some $t > 0$; $a, b, k, t$ integers. True even if it is $\lceil n/b \rceil$ instead of $n/b$.

Expanding recurrence as in earlier examples, to get

$T(n) = n^h \ ( d + c \ S \ )$

where $h = \log_b(a)$ (tree height) and $S = \sum_{j=1}^{\log_b n} x^j$, where $x = b^k/a$.

If $c = 0$ the sum S is irrelevant, and $T(n) = O(n^h)$: all the work happens in the base cases, of which there are $n^h$, one for each leaf in the recursion tree.

If $c > 0$, then the sum matters, and splits into 3 cases (like previous slide):

- if $x < 1$, then $S < x/(1-x) = O(1)$. [S is just the first log n terms of the infinite series with that sum].
- if $x = 1$, then $S = \log_b(n) = O(\log n)$. [all terms in the sum are 1 and there are that many terms].
- if $x > 1$, then $S = x \cdot (x^{1+\log_b(n)}-1)/(x-1)$. After some algebra, $n^h * S = O(n^k)$

Power(a,n)

**Input:** integer $n$ and number $a$

**Output:** $a^n$

Obvious algorithm

$n-1$ multiplications

Observation:

if $n$ is even, $n = 2m$, then $a^n = a^m \cdot a^m$

```
Power(a,n)
   if n = 0 then return(1)
   if n = 1 then return(a)
   x ← Power(a,⌊n/2⌋)
   x ← x•x
   if n is odd then
        x ← a•x
   return(x)
```

Let M(n) be number of multiplies

Worst-case recurrence: $$M(n)=\begin{cases} 0 & n \le 1 \\ M\left(\lfloor n/2 \rfloor\right)+2 & n>1 \end{cases}$$

By master theorem

M(n) = O(log n) (a=1, b=2, k=0)

More precise analysis:

M(n) = $\lfloor \log_2 n \rfloor$ + (# of 1's in n's binary representation) - 1

Time is O(M(n)) if numbers < word size, else also depends on length, multiply algorithm

Instead of $a^n$ want $a^n \bmod N$

$a^{i+j} \bmod N = ((a^i \bmod N) \cdot (a^j \bmod N)) \bmod N$

same algorithm applies with each $x \cdot y$ replaced by

$((x \bmod N) \cdot (y \bmod N)) \bmod N$

In RSA cryptosystem (widely used for security)

need $a^n \bmod N$ where $a, n, N$ each typically have 1024 bits

Power: at most 2048 multiplies of 1024 bit numbers

relatively easy for modern machines

Naive algorithm: $2^{1024}$ multiplies

Idea:

"Two halves are better than a whole"

if the base algorithm has super-linear complexity.

"If a little's good, then more's better"

repeat above, recursively

Analysis: recursion tree or Master Recurrence

Applications: Many.

Binary Search, Merge Sort, (Quicksort), Closest points, Integer multiply, exponentiation,…