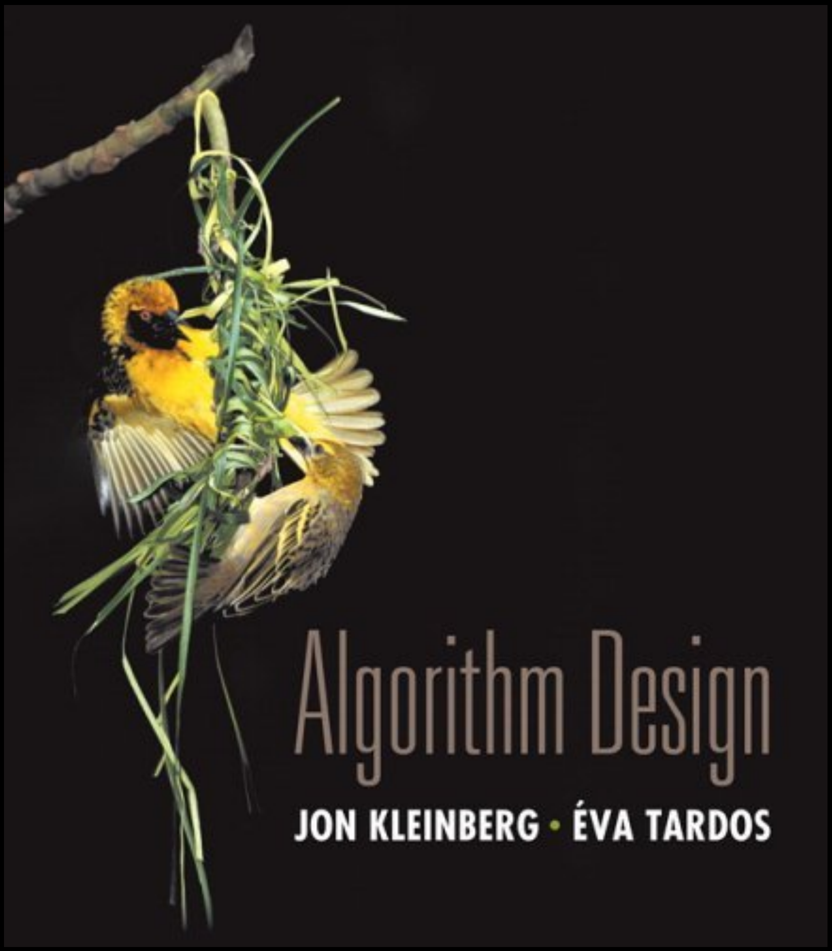


Chapter 5

Divide and Conquer

Addition/modifications by P. Beame, W.L. Ruzzo

5.6 Convolution and FFT

Fast Fourier Transform: Applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, control systems, signal processing, speech recognition, data compression, image processing.
- DVD, JPEG, MP3, MRI, CAT scan.
- Numerical solutions to Poisson's equation.

> The FFT is one of the truly great computational developments of this [20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. *-Charles van Loan*

Fast Fourier Transform: Brief History

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge-König (1924). Laid theoretical groundwork.

Danielson-Lanczos (1942). Efficient algorithm.

Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.

Importance not fully realized until advent of digital computers.

Polynomials: Coefficient Representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add: O(n) arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate: O(n) using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

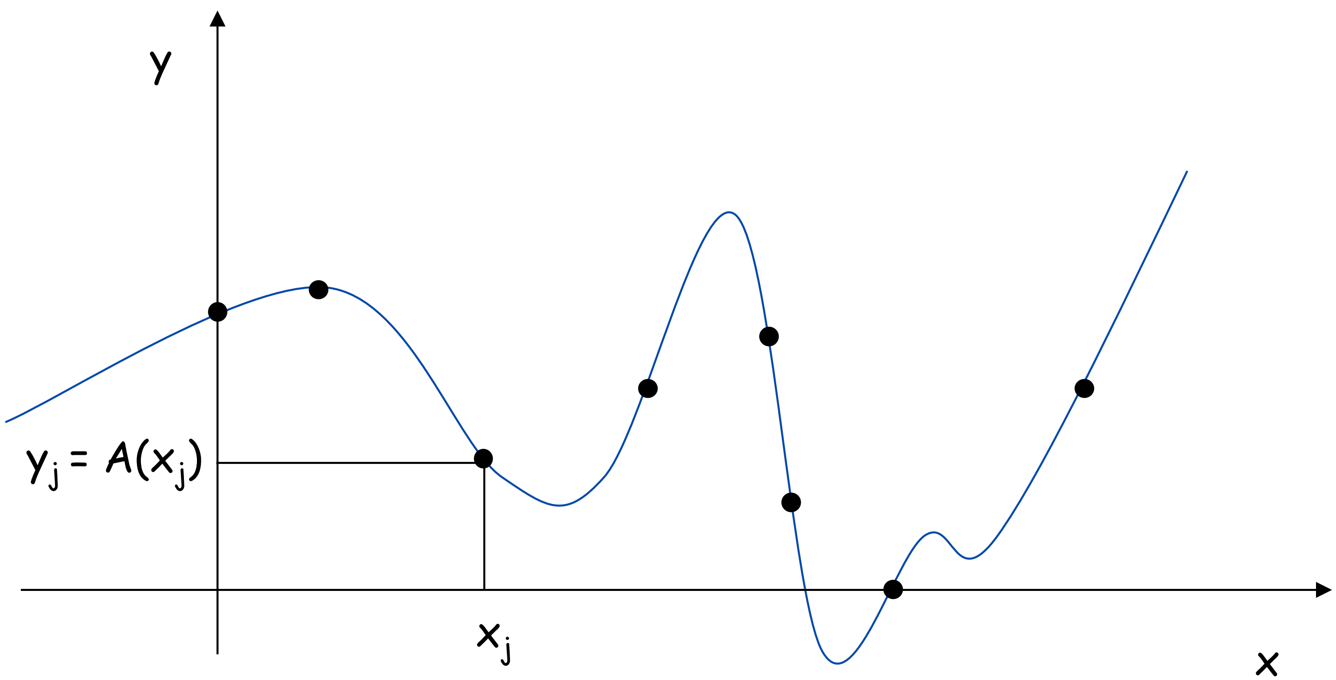
Multiply (convolve): $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Polynomials: Point-Value Representation

Fundamental theorem of algebra. [Gauss, PhD thesis] A degree n polynomial with complex coefficients has n complex roots.

Corollary. A degree n-1 polynomial A(x) is uniquely specified by its evaluation at n distinct values of x.



Polynomials: Operations on Point-Value Representation

Polynomial. [point-value representation]

$$A(x): \ (x_0, y_0), \ \ldots, (x_{n-1}, y_{n-1})$$
$$B(x): \ (x_0, z_0), \ \ldots, (x_{n-1}, z_{n-1})$$

Add: O(n) arithmetic operations.

$$A(x) + B(x): \ \ (x_0, y_0 + z_0), \ldots, (x_{n-1}, y_{n-1} + z_{n-1})$$

Multiply: O(n), but need 2n-1 points.

$$A(x) \times B(x): \ \ (x_0, y_0 \times z_0), \ldots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

Evaluate: $O(n^2)$ using Lagrange's formula.

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting Between Two Polynomial Representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

Representation	Multiply	Evaluate
Coefficient	$O(n^2)$	$O(n)$
Point-value	$O(n)$	$O(n^2)$

Goal. Make all ops fast by efficiently converting between two representations.

$$a_0, a_1, \ldots, a_{n-1} \rightleftarrows (x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$

coefficient representation

point-value representation

Converting Between Two Polynomial Representations: Brute Force

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + ... + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, ..., x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible iff x_i distinct

$O(n^2)$ for matrix-vector multiply

$O(n^3)$ for Gaussian elimination
($O(n^2)$ if you can precompute/save it)

Point-value to coefficient. Given n distinct points $x_0, ..., x_{n-1}$ and values $y_0, ..., y_{n-1}$, find unique polynomial $a_0 + a_1 x + ... + a_{n-1} x^{n-1}$ that has given values at given points.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

Intuition. Choose two points to be ± 1.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.

> Can evaluate polynomial of degree $\le n$ at 2 points by evaluating two polynomials of degree $\le \frac{1}{2}n$ at 1 point.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

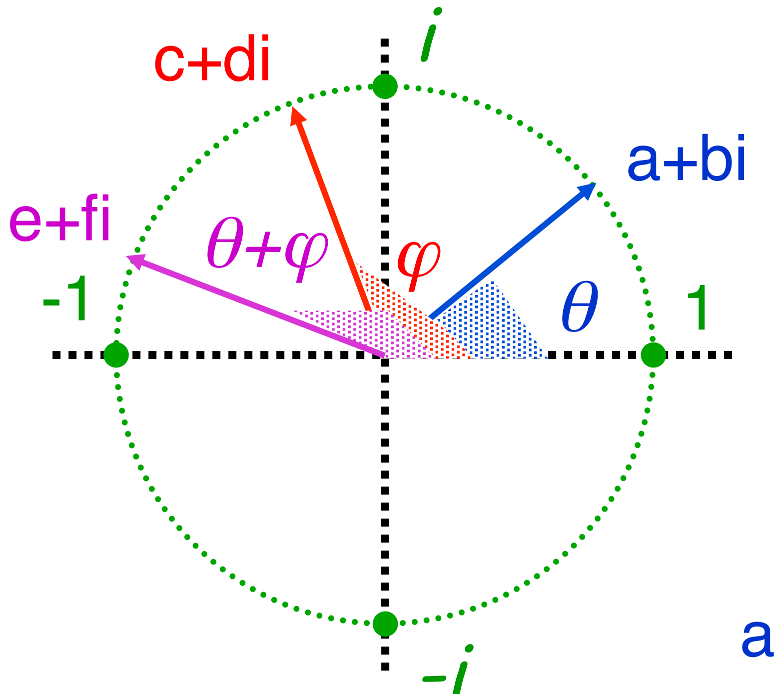
Intuition. Choose four points to be $\pm 1, \pm i$.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.
- $A(i) = A_{even}(-1) + i\, A_{odd}(-1)$.
- $A(-i) = A_{even}(-1) - i\, A_{odd}(-1)$.

Can evaluate polynomial of degree $\leq n$ at 4 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 2 points.

Complex Numbers

$i^2 = -1$



- To add complex numbers, add components (like vectors)
- To multiply complex numbers:
  1. add angles
  2. multiply lengths

  (all lengths = 1 here)

$$e+fi = (a+bi)(c+di)$$

$e^{2\pi i} = 1$

$e^{\pi i} = -1$

$a+bi = \cos\theta + i\sin\theta = e^{i\theta}$

$c+di = \cos\varphi + i\sin\varphi = e^{i\varphi}$

$e+fi = \cos(\theta+\varphi) + i\sin(\theta+\varphi) = e^{i(\theta+\varphi)}$

Discrete Fourier Transform

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Key idea: choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

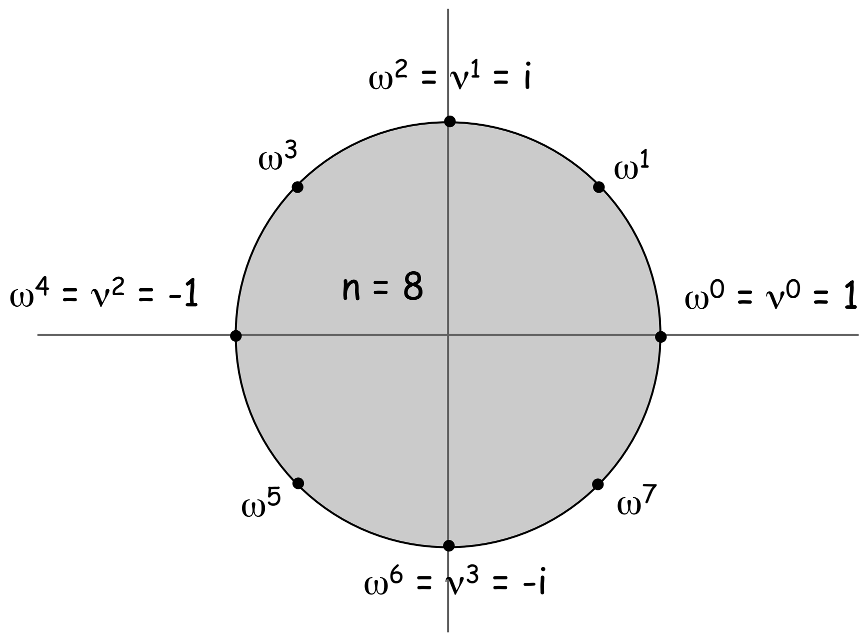
$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{\text{Discrete Fourier transform}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}}_{\text{Fourier matrix } F_n} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Roots of Unity

Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.
Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = e^{4\pi i / n}$.
Fact. $\omega^2 = \nu$ and $(\omega^2)^k = \nu^k$.



Fast Fourier Transform

Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + \ldots + a_{n-1} x^{n-1}$ at n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n/2-2} x^{(n-1)/2}$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n/2-1} x^{(n-1)/2}$.
- $A(x) = A_{even}(x^2) + x \, A_{odd}(x^2)$.

Conquer. Evaluate degree $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}n^{th}$ roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

Combine.

- $A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$
- $A(\omega^{k+n/2}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\omega^{k+n/2} = -\omega^k$

$\nu^k = (\omega^k)^2 = (\omega^{k+n})^2$

FFT Algorithm

$\texttt{fft}(n,\ a_0, a_1, \ldots, a_{n-1})\ \{$

$\quad\quad \textbf{if}\ (n == 1)\ \textbf{return}\ a_0$

$\quad\quad (e_0, e_1, \ldots, e_{n/2-1}) \leftarrow \texttt{FFT}(n/2,\ a_0, a_2, a_4, \ldots, a_{n-2})$

$\quad\quad (d_0, d_1, \ldots, d_{n/2-1}) \leftarrow \texttt{FFT}(n/2,\ a_1, a_3, a_5, \ldots, a_{n-1})$

$\quad\quad \textbf{for}\ k = 0\ \textbf{to}\ n/2 - 1\ \{$

$\quad\quad\quad\quad \omega^k \leftarrow e^{2\pi i k/n}$

$\quad\quad\quad\quad y_k \leftarrow e_k + \omega^k\ d_k$

$\quad\quad\quad\quad y_{k+n/2} \leftarrow e_k - \omega^k\ d_k$

$\quad\quad \}$

$\quad\quad \textbf{return}\ (y_0, y_1, \ldots, y_{n-1})$

$\}$

FFT Summary

Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the n^{th} roots of unity in O(n log n) steps.

assumes n is a power of 2

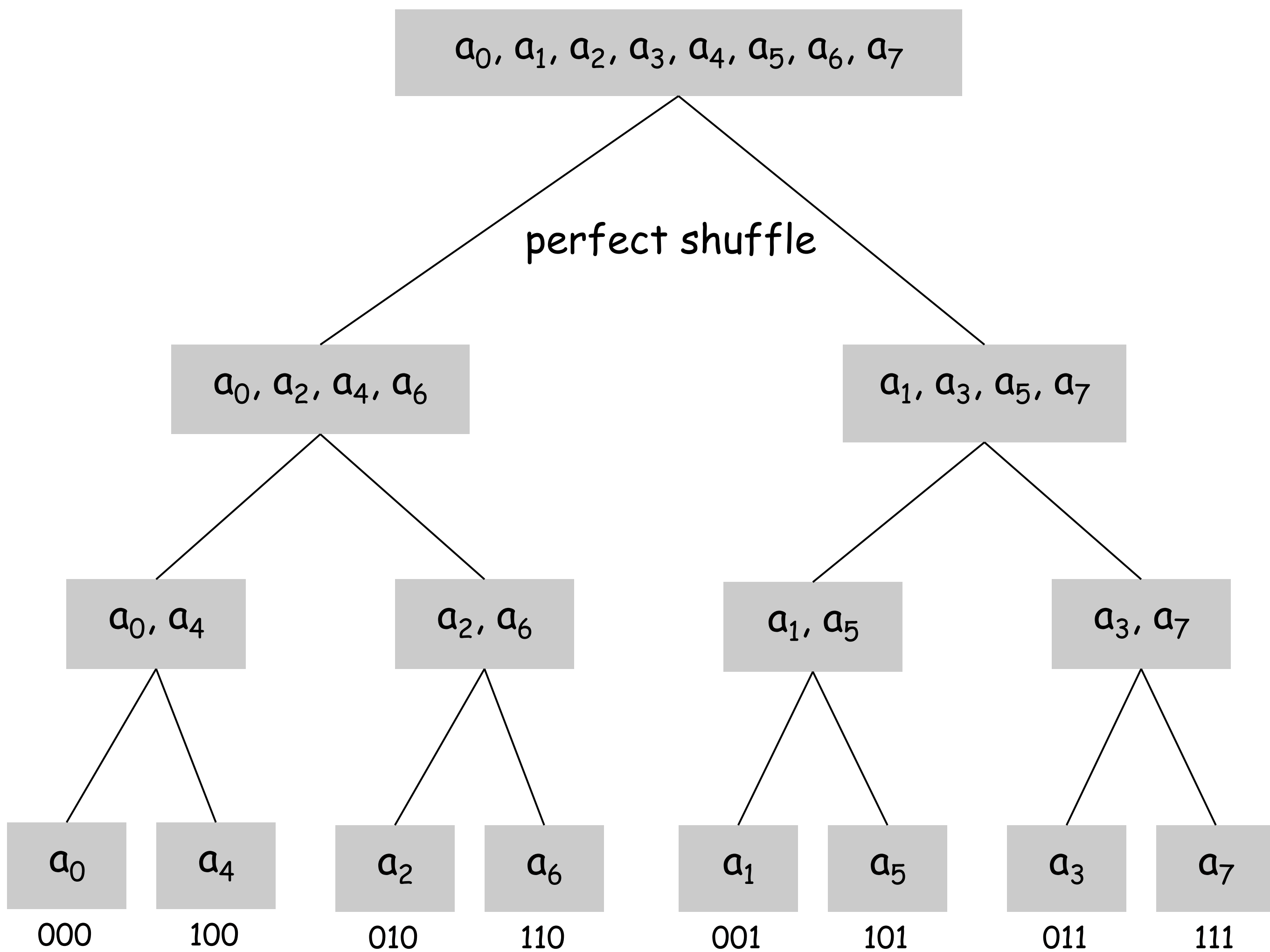
Running time. $T(2n) = 2T(n) + O(n) \Rightarrow T(n) = O(n \log n)$.

$$a_0, a_1, \ldots, a_{n-1} \underset{}{\overset{O(n \log n)}{\rightleftarrows}} (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$$

coefficient representation

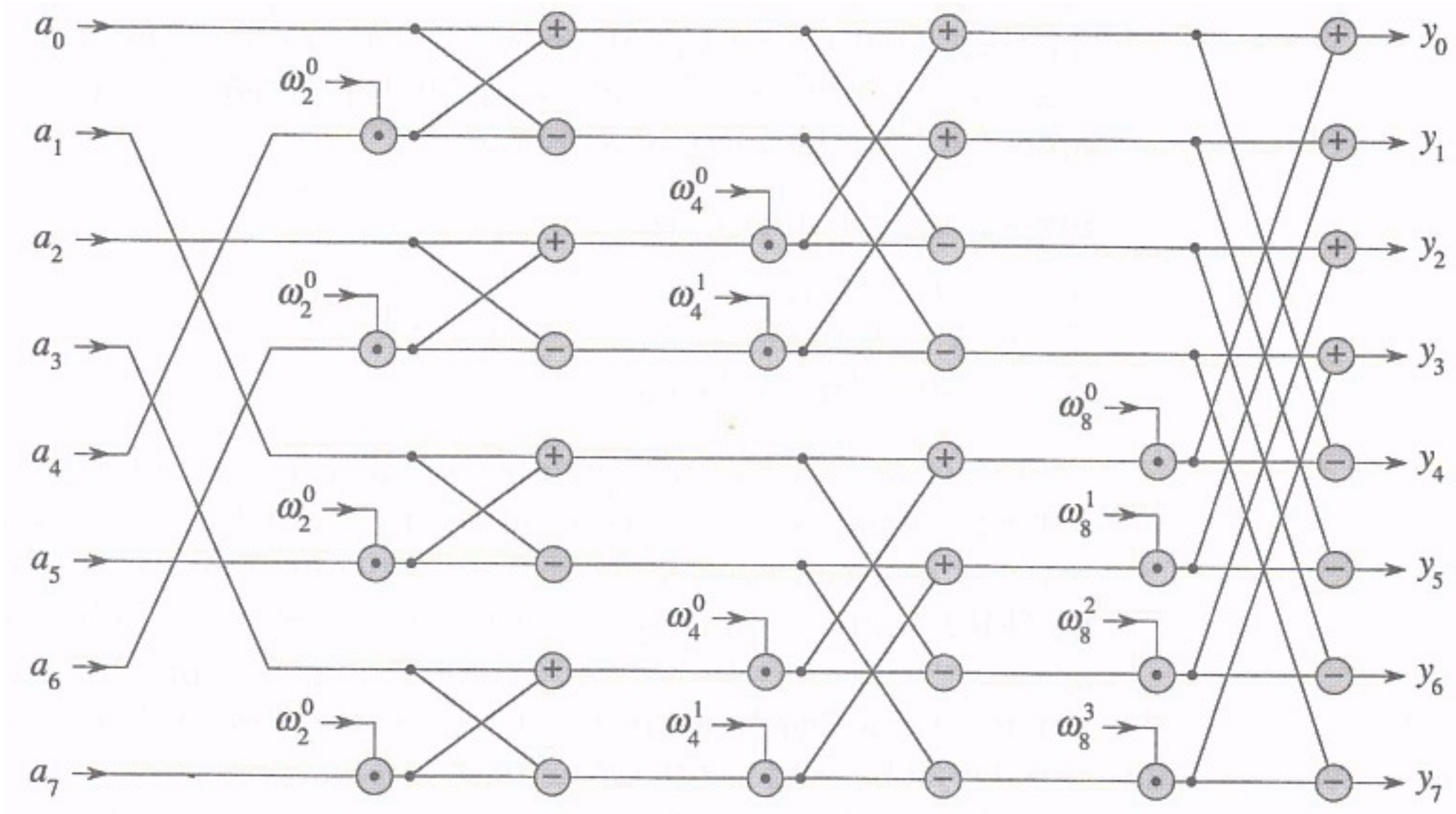
point-value representation

Recursion Tree



"bit-reversed" order

Non-Recursive View



$\omega_n = e^{2\pi i/n}$

CLR 1990, fig 32.5 pg 796

Point-Value to Coefficient Representation: Inverse DFT

Goal. Given the values $y_0, \ldots, y_{n-1}$ of a degree n-1 polynomial at the n points $\omega^0, \omega^1, \ldots, \omega^{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Inverse DFT

Fourier matrix inverse $(F_n)^{-1}$

Inverse FFT

Claim. Inverse of Fourier matrix is given by following formula.

$$G_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide by n).

Inverse FFT: Proof of Correctness

Claim. F_n and G_n are inverses.

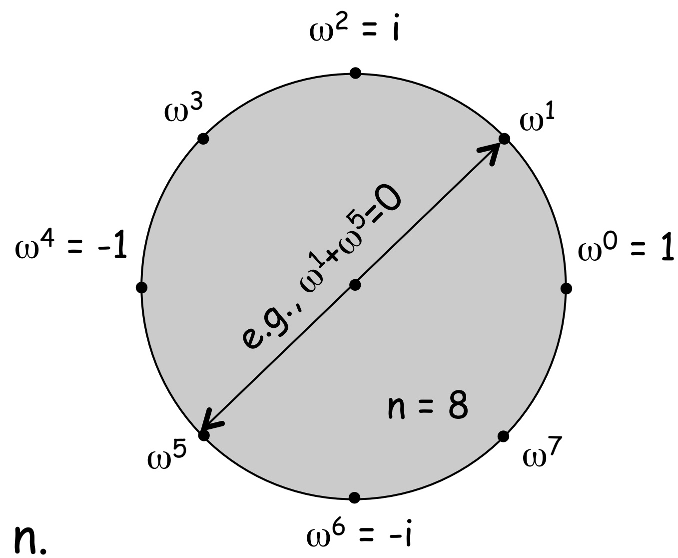
Pf.

$$\left(F_n\, G_n\right)_{k\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{kj}\, \omega^{-jk'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')\,j} \underset{\uparrow}{=} \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$



Pf.

- If k is a multiple of n then $\omega^k = 1 \Rightarrow$ sums to n.
- Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \Rightarrow$ sums to 0. ▪

Inverse FFT: Algorithm

```
ifft(n, a_0,a_1,…,a_{n-1}) {
   if (n == 1) return a_0

   (e_0,e_1,…,e_{n/2-1}) ← FFT(n/2, a_0,a_2,a_4,…,a_{n-2})
   (d_0,d_1,…,d_{n/2-1}) ← FFT(n/2, a_1,a_3,a_5,…,a_{n-1})

   for k = 0 to n/2 - 1 {
      ω^k ← e^{-2πik/n}
      y_k       ← (e_k + ω^k d_k) / n
      y_{k+n/2} ← (e_k - ω^k d_k) / n
   }

   return (y_0,y_1,…,y_{n-1})
}
```

Inverse FFT Summary

Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the n^{th} roots of unity in O(n log n) steps.

assumes n is a power of 2

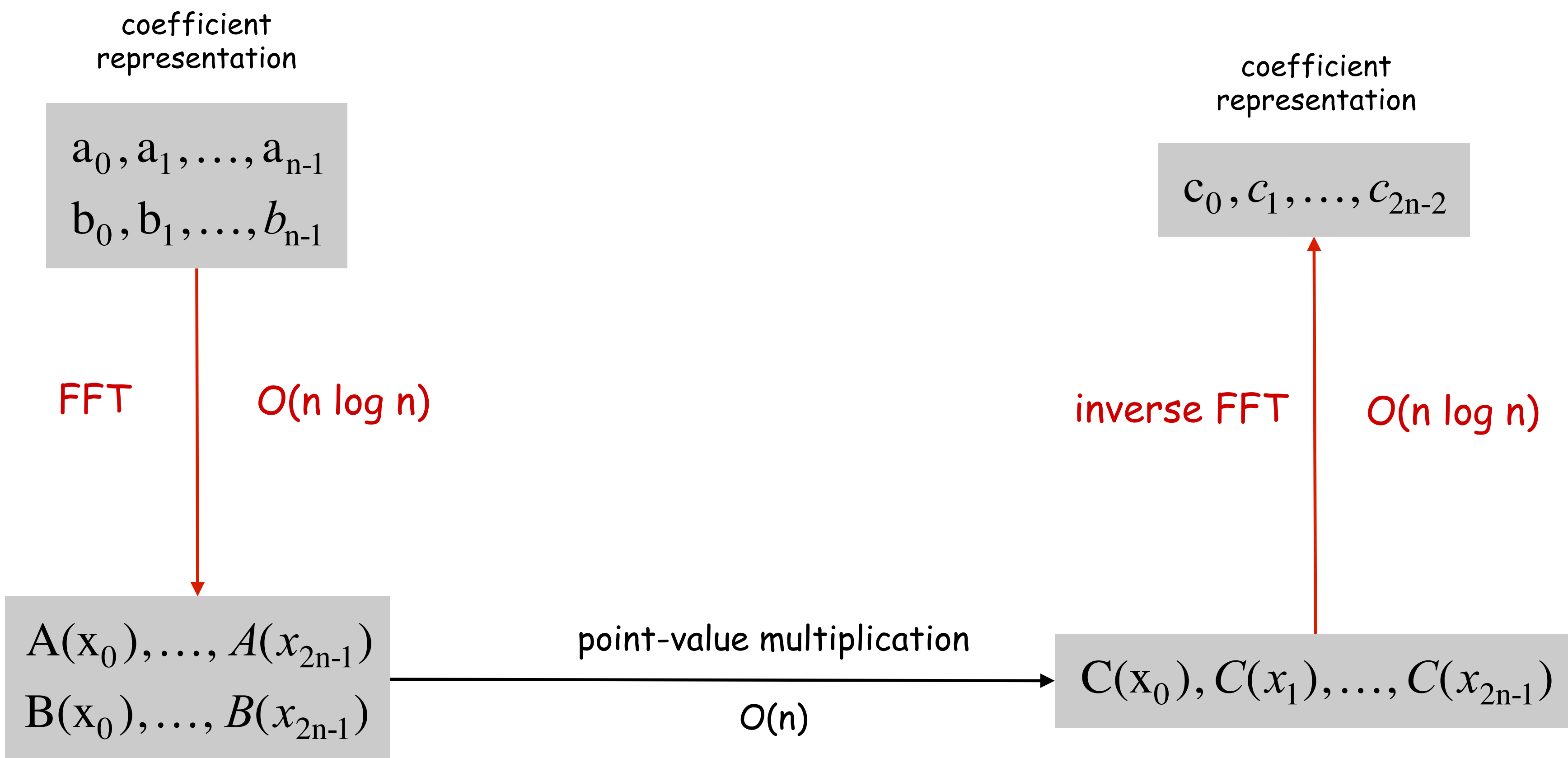
$$\begin{tikzcd}[column sep=huge] a_0, a_1, \ldots, a_{n-1} \arrow[r, shift left, "O(n \log n)"] & (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1}) \arrow[l, shift left, "O(n \log n)"] \end{tikzcd}$$

coefficient representation

point-value representation

Polynomial Multiplication

Theorem. Can multiply two degree n-1 polynomials in O(n log n) steps.



FFT in Practice

Fastest Fourier transform in the West. [Frigo and Johnson]

- Optimized C library.
- Features: DFT, DCT, real, complex, any size, any dimension.
- Won 1999 Wilkinson Prize for Numerical Software.
- Portable, competitive with vendor-tuned code.

Implementation details.

- Instead of executing predetermined algorithm, it evaluates your hardware and uses a special-purpose compiler to generate an optimized algorithm catered to "shape" of the problem.
- Core algorithm is nonrecursive version of Cooley-Tukey radix 2 FFT.
- O(n log n), even for prime sizes.

Reference: http://www.fftw.org

Integer Multiplication

Integer multiplication. Given two n bit integers $a = a_{n-1} \ldots a_1 a_0$ and $b = b_{n-1} \ldots b_1 b_0$, compute their product $c = a \times b$.

Convolution algorithm.

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

- Form two polynomials.
- Note: $a = A(2)$, $b = B(2)$.
- Compute $C(x) = A(x) \times B(x)$.
- Evaluate $C(2) = a \times b$.
- Running time: $O(n \log n)$ complex arithmetic steps.

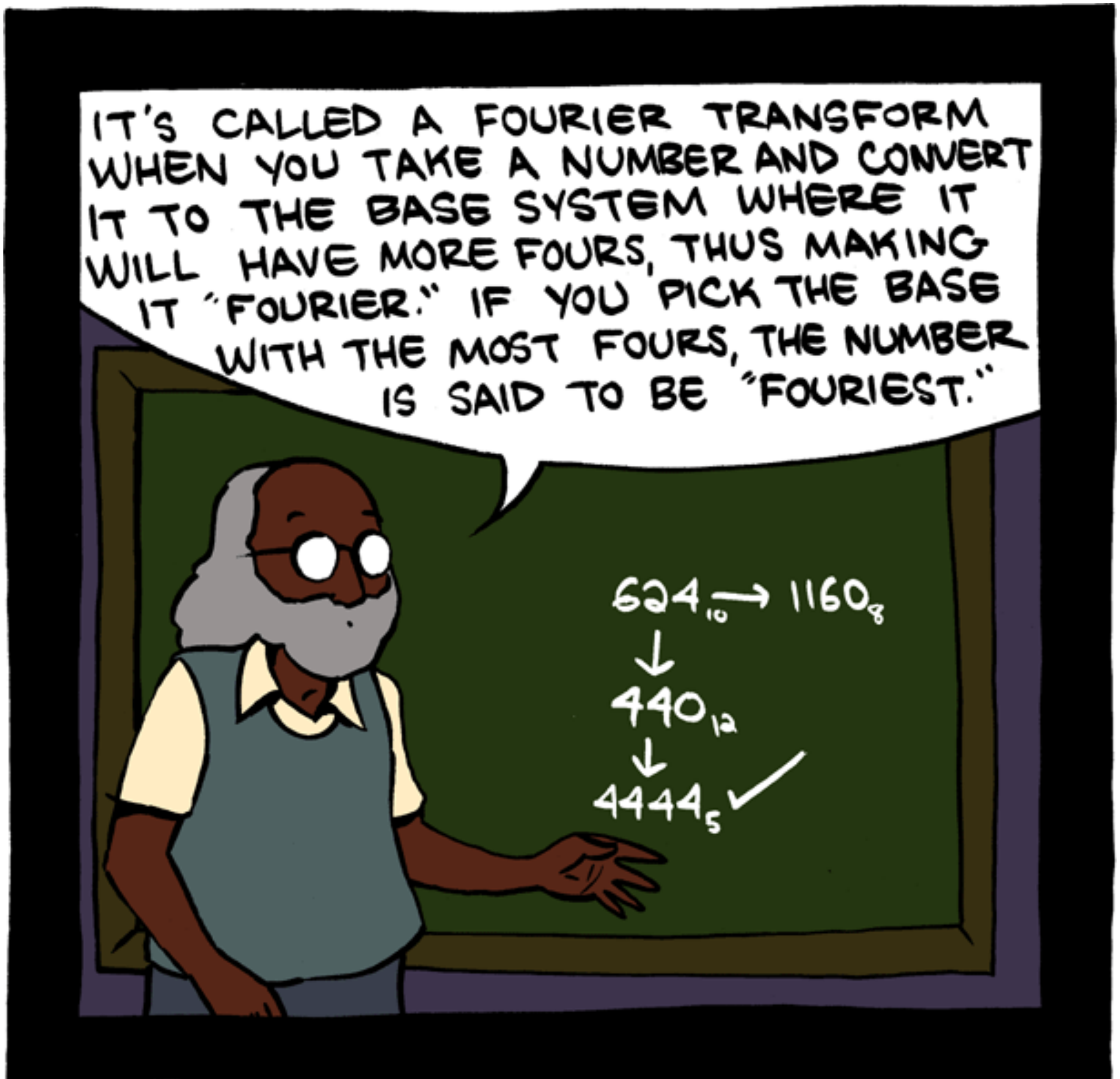
Theory.

[Schönhage-Strassen 1971] $O(n \log n \log \log n)$ bit operations.

[Fürer, 2007] $(n \log n)\, 2^{O(\log^* n)}$

[NB: $\log^* n \leq 5$ for all practical purposes, but the big-O is nasty.]

Practice. [GNU Multiple Precision Arithmetic Library] GMP proclaims to be "the fastest bignum library on the planet." It uses brute force, Karatsuba, and FFT, depending on the size of n.



Teaching math was way more fun after tenure.

http://www.smbc-comics.com/comics/20130201.gif