The Economics of Social Networks

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Abstract

The science of social networks is a central field of sociological study, a major application of random graph theory, and an emerging area of study by economists, statistical physicists and computer scientists. While these literatures are (slowly) becoming aware of each other, and on occasion drawing from one another, they are still largely distinct in their methods, interests, and goals. Here, my aim is to provide some perspective on the research from these literatures, with a focus on the formal modeling of social networks and the two major types of models: those based on random graphs and those based on game theoretic reasoning. I highlight some of the strengths, weaknesses, and potential synergies between these two network modeling approaches.

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1 Introduction

Social networks are the fabric of many of our interactions. Such networks include the relationships among friends and relatives with whom we share information and favors on a regular basis, and reach as far as influencing decisions by many of the world's companies regarding with whom and how they conduct their business. The many regularities in network structure across applications makes a scientific study of social networks a possibility. The deep and pervasive impact that networks have on behavior makes such a study a necessity.

The science of social networks was initiated by sociologists more than a century ago, and has grown to be a central field of sociological study over the past fifty years.[1] Over that same period, a mathematical literature on the structure of random graphs moved steadily along, but with intermittent but surprisingly few ties to the sociological literature.[2] While economists have occasionally showed interest in networks, an explosion of studies of networks using game-theoretic modeling techniques and with economic perspectives has occurred over the last decade.[3] A recent awakening of an interest in social networks has also occurred in the computer science and statistical physics literatures, mainly over the past five or six years.[4] While these literatures are (slowly) becoming aware of each other, and on occasion drawing from one another, they are still largely distinct in their methods, interests, and approaches. My goal here is to provide some perspective on the research from these literatures, with a focus on the formal modeling of social networks, and to highlight some of the strengths, weaknesses, and potential synergies between the two main approaches.

Given the breadth of these combined literatures, and the fact that there are surveys covering the various literatures,[5] my aim here is not to try to give a comprehensive overview of the literatures, but rather to try to put some of the main contributions and techniques of formal modeling of social networks in context and to relate them to each other. I focus on two main threads of the literatures: the first is models of the formation of networks and the

[1] See Freeman (2004) for some history of thought of the sociology literature.

[2] See Bollobás (2001) for a survey of the random graph literature.

[3] The books Dutta and Jackson (2003) and Demange and Wooders (2004) contain various surveys and papers.

[4] See Newman (2003).

[5] The sociology literature is too vast for any article to adequately survey, but introductory texts, such as Wasserman and Faust (1994), as well as the recent history of thought book by Freeman (2004), are useful starting points. Concerning the economics literature, see Jackson (2003, 2004) for strategic modeling of networks; van den Nouweland (2004) for graphs and networks in cooperative game theory; Goyal (2004) for learning on networks; Ioannides and Datcher-Loury (2004) for networks in labor economics; Page and Kamat (2004) for farsighted formation of networks; and Bloch (2004) for networks in industrial organization. See Newman (2003, 2004) for surveys covering some of the recent statistical physics and part of the computer science literatures. There are also books that touch on some parts of the physics literature, such as Watts (1999), Barabasi (2002), Vega-Redondo (2004). A text that bridges some of the modeling from the various literatures is by Jackson (2005).

second is models of how social behavior and economic outcomes are influenced by network structure.

In order to provide some context, I start by giving some basic background on social networks and a very cursory look at a few things that have been learned from empirical studies. Next, I turn to discuss models of formation of networks. Here, I distinguish between two different approaches that have been taken. One has its roots in the random graph literature and models formation by specifying either some stochastic process or an algorithmic process through which the links in a network are formed. This literature has mainly focussed on deducing properties of large networks. The second approach is game theoretic and stems from the economics literature. It has mainly focussed on models where the links are formed at the discretion of the nodes who derive benefits and face costs associated with various links and network configurations. These two approaches lead to very complementary insights regarding networks, each of which is adapted to answering different sorts of questions. They also have different strengths and weaknesses that I highlight. Finally, I discuss models that take network structure as a given and study the influence that networks have on social and/or economic outcomes. This last area of study shows why the science of social networks is important for more than just an understanding of the networks themselves.

2 Some Background on Networks

The systematic study of social networks by sociologists dates from the 1920's and 30's, took root in the 1960's, and has grown rapidly over the past four decades.[6] That literature includes many case studies from which has emerged a rich mosaic of characteristics that are shared by many social networks, as well as a taxonomy for measuring and describing social networks and a broad set of hypotheses and theories about network form and influence. Much of what I discuss in this section is either directly from that literature, or was influenced by it.

2.1 Some Examples of Social Networks

To fix ideas, let me begin by presenting two networks that have been studied in some detail. This gives an idea of various applications, as well as a glimpse of some of the characteristics that have been studied.

Example **1** *Florentine Marriages*

The first example is a network analyzed by Padgett and Ansell (1993). It is the network of marriages between the key families in Florence in the 1430's. The following figure provides

[6] Again, see Freeman (2004) for some history of thought. Interestingly, while Freeman laments the disconnect between the traditional sociology literature and the emerging physics literature on networks, the gulf between the sociology and economics literatures seems to be at least as large.

the links between the key families in Florence at that time, where a link represents a marriage between members of the two linked families.[7]

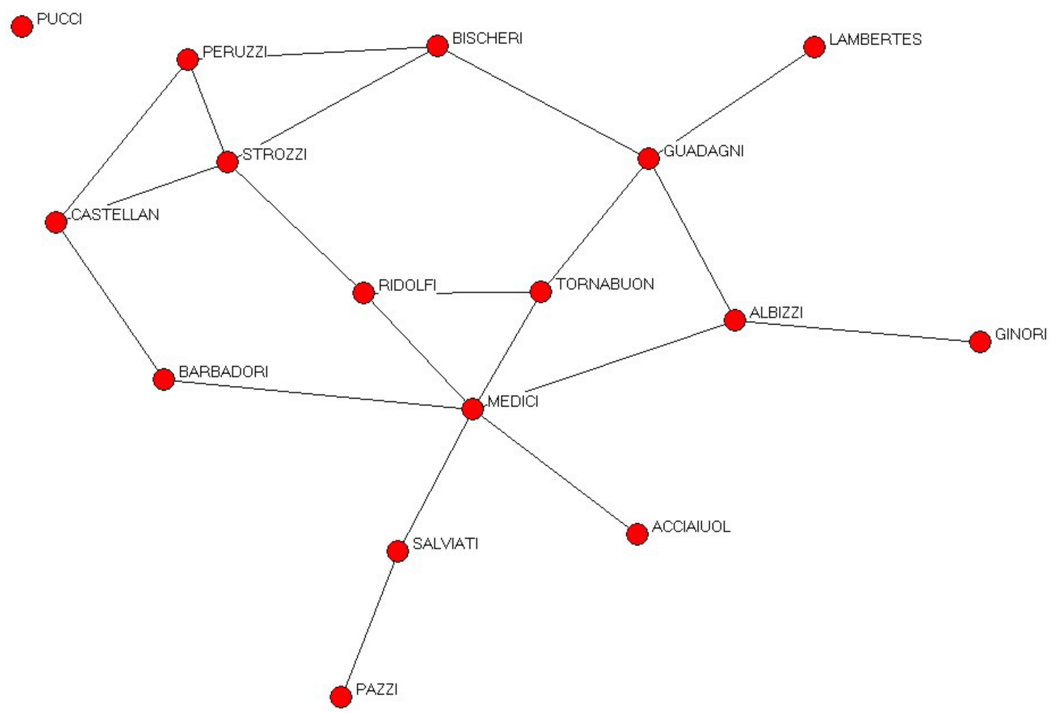


Figure 1: 15th Century Florentine Marriges (Padgett and Ansell (1993))

As Padgett and Ansell (1993) explain, during this time period the Medici (with Cosimo de' Medici playing the key role) rose in power and largely consolidated control of the business and politics of Florence. Previously Florence had been ruled by an oligarchy of elite families. If one examines wealth and political clout, however, the Medici did not stand out at this point and so one has to look at the structure of social relationships to understand why it was the Medici who rose in power. For instance, the Strozzi had both greater wealth and seats in the local legislature, and yet the Medici rose to eclipse them. A key to understanding this, as Padgett and Ansell (1993) detail, can be seen in the network structure.

Let $P(ij)$ denote the number of shortest paths connecting family i to family j.[8] Let $P_k(ij)$ denote the number of these shortest paths between i and j that include k as a node on one of the links. Now in order to get a fuller feel for how central families are, we can look at an average of this betweeness calculation. Averaging across all pairs of other families and

[7]The data here were originally collected by Kent [212], but were first coded by Padgett and Ansell (1993) who discuss the network relationships in more detail.

[8]For formal definitions of paths, distances, and related notions, see below.

normalizing by the number of possible pairs gives us a betweeness measure (due to Freeman (1977)) for a given family. In particular, betweeness centrality is defined as $\sum_{ij} \frac{P_k(ij)/P(ij)}{(n-1)(n-2)/2}$ for each family k. This measure of betweeness for the Medici is .522. That means that if we look at all the shortest paths between various families (other than the Medici) in this network, the Medici lie on over half of them![9] In contrast, a similar calculation for the Strozzi comes out at .103, or just over ten percent. The second highest family in terms of betweeness after the Medici is the Guadagni with a betweeness of .255. To the extent that marriage relationships were keys to communicating information, business deals, and reaching political decisions, the Medici were much better positioned than other families, at least according to this notion of betweeness. While aided by circumstance (for instance, fiscal problems resulting from wars), it was the Medici and not some other family that ended up consolidating power. As Padgett and Ansell (1993) explain, "Medician political control was produced by network disjunctures within the elite, which the Medici alone spanned."

It should be emphasized that the Medici came to have such a special position in the network through careful planning. As Padgett and Ansell [280] say (footnote 13), "The modern reader may need reminding that all of the elite marriages recorded here were arranged by patriarchs (or their equivalents) in the two families. Intraelite marriages were conceived of partially in political alliance terms." Thus, in order to understand how this network, and not some other network, came to arise it is important to have models of strategic network formation, a theme that we shall return to.

2.2 Some Notation

With one example of a social network under our belt, let me now introduce some basic definitions, as these will be useful in discussing the second example and what follows thereafter.

Let $N = \{1, 2, \ldots, n\}$ denote a set of *nodes*, which represent the social agents who might be tied in a network of social relationships. In the example above, these are the Florentine families. In the next example these are individual people (researchers), and in other examples they might be firms, web pages, countries, etc.

A *network* g can be represented by an $n \times n$ matrix taking on values 0 or 1. The idea is that if $g_{ij} = 1$, then i is linked to j. In various applications, it might be that these links are undirected, as in the Florentine families example where marriage is a reciprocal relationship. In such settings $g_{ij} = g_{ji}$ by necessity. In other applications, such as an example where a link represents a citation of one research article by another, the network is naturally directed. In such cases, it is possible that $g_{ij} = 1 \neq 0 = g_{ji}$.[10]

[9] This network only codes a set of the key families at the time, and the full data set includes many more marriages and families.

[10] This is a basic, but still very useful and standard way of encoding networks. In some applications, the

For simplicity in notation, I write ij to represent the *link* between i and j, and also write $ij \in g$ to indicate that i and j are linked under the network g. Shorthand notations for the network obtained by adding or deleting a link ij to or from an existing network g are $g + ij$ and $g - ij$, respectively.

For any network g and agent or node i, let $N_i(g)$ be the *neighborhood* of i in g, that is, the set of agents linked to i in the network g, so that $N_i(g) = \{j \mid ij \in g\}$.

A *path* in a network $g \in G$ between agents i and j is a sequence of agents $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \dots, K-1\}$, with $i_1 = i$ and $i_K = j$. The *length* of such a path is $K - 1$, the number of links involved.[11]

A *component* of a network is a maximal connected subgraph. That is, g' is a component of g if: (a) g' is a subnetwork of g (so $ij \in g'$ only if $ij \in g$), (b) $ij \in g'$ and $k\ell \in g'$ implies that there is a path between i and k in g', and (c) $ij \in g'$ and $ik \in g$ implies $ik \in g'$. The network pictured in Figure 3 has two components, one consisting of the isolated node 25, and the other consisting of the graph between nodes 1 to 24.

The *distance* between two nodes i and j, denoted $d(i, j)$, is the minimum path length between i and j (and set to be infinite if no such path exists).

The *diameter* of a network g is defined as $\bar{d}(g) = \max_{i,j} d(i, j)$, the maximum distance between any two nodes. If a network is not connected (there are at least two nodes that have no path between them), then the diameter is infinite. As many social networks are not connected, the diameter is often reported for the largest component. For example, in Figure 1, the network is not connected as the Pucci are isolated, and the diameter of the largest component is 5 (the distance from the Pazzi to the Lambertes or the Pazzi to the Peruzzi).

Another characteristic of networks is referred to under a variety of names including cliquishness, transitivity, and *clustering*. While there are many variations, the basic idea is to measure how dense the network is on a very local level. Given a node, what fraction of that node's friends or neighbors are friends or neighbors of each other? In particular, if i has links to both j and k, are j and k linked to each other?[12] The percentage of times that the answer is "yes" with regard to a node i is i's clustering coefficient. One can then average

strength of a link or some other aspect of link may be important, or there may be different types of links that can be simultaneously held between nodes. For the purposes of this article, I will stick with the basic model.

[11] In the case of directed networks, one can keep track of directed paths as well as undirected ones. I will be explicit when necessary, and otherwise assume that links are treated as if they are not directed.

[12] For a directed network, one can either treat links as if they are undirected, or else can look for cycles (when directed links ij and jk are present, one counts the percent of ki's).

across all nodes in the network. Thus the clustering for a node i is[13]

$$C_i(g) = \frac{\#\{jk \in g \mid k \neq j,\, j \in N_i(g),\, k \in N_i(g)\}}{\#\{jk \mid k \neq j,\, j \in N_i(g),\, k \in N_i(g)\}}.$$

In Figure 1, the clustering for the Medici is 1/15, for the Bisteri is 1/3, and for the Guadagni is 0. The average clustering coefficient is[14]

$$C^{avg}(g) = \sum_i \frac{C_i(g)}{n}.$$

EXAMPLE **2** *Erdös Numbers and Co-authorship Networks Among Researchers*

With some definitions in hand, let us turn to another example. Our next example concerns much larger networks. These are networks that keep track of collaboration among researchers. Here a link represents the co-authorship of a paper during some time period covered by the study. The well-known and prolific mathematician Paul Erdös had many co-authors, and as a fun distraction many mathematicians (and economists for that matter) have found the shortest path(s) from themselves to Erdös. [15] These networks are also of scientific interest themselves, as they tell us something about how research is conducted and also how information and innovation might be disseminated. Such studies have now been conducted in various fields, including mathematics (Grossman and Ion (1995), de Castro and Grossman (1999)), biology and physics (Newman (2001, 2002)), and economics (Goyal, van der Leij and Moraga-González (2003)). Various statistics from these studies give us some impression of the network structure.[16]

One interesting feature of the networks concerns the path lengths. Here we see that despite the noncomparabilities of the networks along many dimensions, average path length and diameters of each of the networks are very comparable. Moreover, these are of an order

[13] If the node i has fewer than two neighbors so that the denominator is of $C_i(g)$ is 0, then one can adopt the convention of setting $C_i(g) = 1$. When averaging across i to determine average clustering, such a convention can make a difference and so it makes sense to ignore nodes that have fewer than two neighbors.

[14] Note that this weights the calculations by averaging across nodes rather than links. That is, a node that has just two neighbors is weighted the same as a node that has two hundred neighbors, even though the second node accounts for many more potential triangles in the network. An alternative measure simply examines the number of times the link ik is present over all combinations of pairs of links ij and ik in the network, and divides by the number of pairs of links present in the network. The difference between these two measures can be quite substantial.

[15] A web site (www.oakland.edu/enp/) maintained by Jerry Grossman, Patrick Ion, and Rodrigo de Castro provides a part of that graph. There are similar networks that have been mapped out for other relationships, such as the Kevin Bacon network (see the web site at the computer science department at the University of Virginia, www.cs.virginia.edu/oracle/), where a link indicates that two actors appeared in the same movie.

[16] As these networks are not connected (there are many isolated authors), the figures for average path length and diameter are reported for the largest component.

Table 1: Co-Authorship Networks

	Biology	Economics	Math	Physics
number of nodes	1520521	81217	253339	52909
avg. degree	15.5	1.7	3.9	9.3
avg. path length	4.9	9.5	7.6	6.2
diameter	24	29	27	20
clustering	.09	.16	.15	.45
% size largest component	.92	.41	.82	.85

substantially smaller than the number of nodes in the network. This is an aspect of the "small-world" nature of social networks that I shall also discuss shortly.

The degree is the most basic characteristic of a node - it represents the number of links that each node has. The average degree varies greatly across these fields. However, we need to be careful in comparing the degrees across studies, as the studies compute total number of co-authors over a period of time, and the time periods differ across studies. Also, the number of co-authors per paper varies dramatically across fields. For instance, in the economics data set, there are on average 1.6 authors per paper (and only 12 percent have more than two authors), while in biology there are on average 3.8 authors per paper. Papers also differ greatly in length across fields leading to different numbers of papers and hence collaborators per unit of time.

A more interesting comparison of degrees is across nodes within a given network. There tends to be a wide range of degrees across nodes. We can get some impression by examining the full distribution of node degrees. Figure 2 provides a log-log plot of the distribution of degrees from the economics co-authorship data from Goyal, van der Leij, Moraga-Gonzalez (2003).

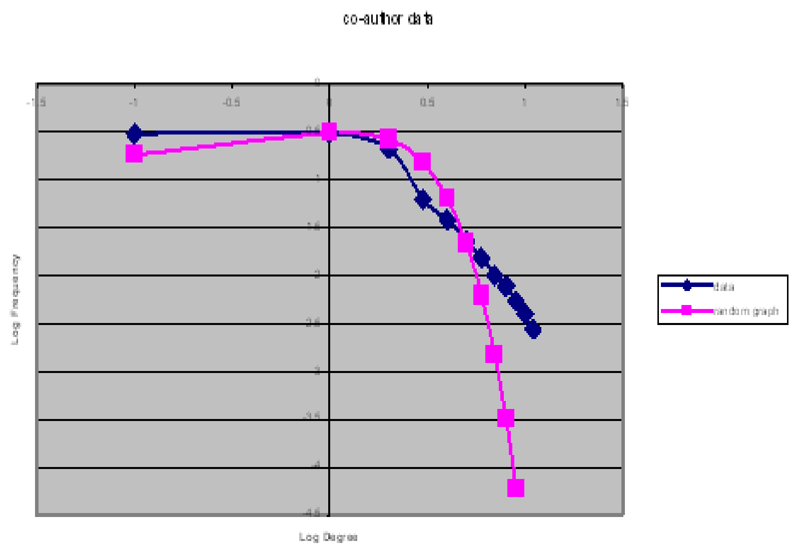


Figure 2: Comparison of the Degree Distributions of a Co-Authorship Network and a (Bernoulli) Random Graph with the same Average Degree

The degrees of economists in the data set range from 0 to over 50. The distribution also has an interesting shape. It clearly exhibits some curvature. However it also shows "less" curvature than the distribution of degrees generated from a network with the same number of links, but where the links are chosen uniformly at random (termed a Bernoulli random graph, discussed below). What this indicates is that there are more economists with very high degree and more with very low degree than we would see in a network where links were generated uniformly at random. This "fat-tailed" property is one that I discuss in more detail below.

2.3 The Prevalence of Network Interactions

While the examples in the previous section give us an idea of the variety of networks that have been studied, it is also important for us to have an idea of what role networks might play in a society and how they might influence economic outcomes. The most obvious and perhaps pervasive role of networks is as a conduit of information, and one of the most extensively documented role for social networks in economics is that of contacts in labor markets.[17]

The magnitude of use of social contacts as a method of matching workers and firms can be seen from various studies. One of the earliest studies, Myers and Shultz (1951), was

[17]For a recent comprehensive overview of research on networks in labor markets see Ioannides and Loury (2004).

based on interviews with textile workers and found that 62 percent had found their first job through a social contact, in contrast with only 23 percent who applied by direct application, and the remaining 15 percent who found their job through an agency, ads, etc. A study by Rees and Shultz (1970) showed that these numbers were not peculiar to textile workers, but applied very broadly. For instance, the percentage of those interviewed who found their jobs through the use of social contacts as a function of their profession was: typist - 37.3 percent, accountant - 23.5 percent, material handler - 73.8 percent, janitor - 65.5 percent, and electrician - 57.4 percent. Moreover, the prevalent use of social contacts in finding jobs is robust across race and gender (See Corcoran, Datcher, and Duncan (1980)) and across country (see Pellizzari (2004)).

The role of social networks is not unique to labor markets, but has been documented much more extensively. For example, networks and social interactions play a role in crime,[18] in trade,[19] social insurance,[20], as well as disease transmission, language and culture, and interactions of firms.

2.4 Some Basic Characteristics of Social Networks

Beyond the fact that social networks play a role in many interactions, we also know a great deal about some basic characteristics of social networks.

2.4.1 Small Worlds

One of the most influential studies of social networks was Stanley Milgram's (1967) ingenious "small-worlds" experiment. Milgram gave booklets with instructions to individuals in one place (Nebraska, in the original experiment). The objective was to get the booklet to a geographically distant individual (on the east coast), where the sender is given some information about the target (e.g., the person's name, occupation, and where they live). The key was that the subjects could only send the booklet to an acquaintance. The acquaintance could then forward the letter to another acquaintance, with the same objective of having the booklet eventually reach the target. The experiment collected information regarding the full chain that the booklets followed, including demographic information about each of the acquaintances along the route. One remarkable statistic was that roughly a quarter of the booklets

[18] Reiss (1980, 1988) finds that two thirds of criminals commit crimes with others, and Glaeser, Sacerdote and Scheinkman (1996) find that social interaction is important in determining criminal activity, especially with respect to petty crime, youth activity in crime, and in areas with less intact households.

[19] Uzzi (1996) finds that relation specific knowledge is critical in the garment industry and that social networks play a key role in that industry. Weisbuch, Kirman, Herreiner (2000) study repeated interactions in the Marseille fish market and discuss the importance of the network structure.

[20] Fafchamps and Lund (2003) show that social networks are critical to the understanding of risk-sharing in rural Philippines, and De Weerdt (2002) provides similar analyses in Africa.

reached their destination.[21] Of the chains that were successful, the maximum number of links that a booklet took was 12 and the median was 5! Given that these would generally not have taken the shortest routes from initial sender to target (as the senders are often not fully aware of the most efficient path to the target), these numbers were quite striking. Since Milgram's initial experiment, this sort of study has been widely replicated and extended, finding similarly short paths and many other interesting facts about the sorts of paths that are followed and the strategies that senders employ to find a shortest route.[22]

A simple back-of-the-envelope calculation gives some insight into this. If most individuals in the world have hundreds of acquaintances, then starting from a given individual, the network size (in terms of number of individuals reached) will expand by a factor on the order of a hundred raised to the power of the path length. It will not take very long paths until the network is the size of the whole world's population. If the network were a tree,[23] simple variations on this method would allow us to calculate average path lengths in a network quite easily. However, most social networks are not trees and so to do a proper calculation, one needs to account for overlap in neighborhoods. Since the percentage of paths which are reaching new individuals decreases (nonlinearly) with the path length, precise average path-length calculations can be very difficult even in fairly easily described networks.[24] Nonetheless, this gives us a feeling for why the diameter of a social network will tend to be much smaller than the number of nodes.

Milgram's finding of a short average path length is echoed in studies of a variety of networks. Many social networks tend to have small diameter and small average path length, where small is on the order of the log of the number of nodes or less.[25] We saw this in the co-authorship networks above. We see this in a network studied by Watts and Strogatz [354], who report a mean distance 3.7 in a network among actors where a link indicates that two actors have been in a movie together. A particularly striking example is that of the World Wide Web. Adamic (1999) reports on a data set collected by Pitkow who finds a mean path length of 3.1 in a network of links among 50 million web pages.

2.4.2 "High" Clustering

While it is interesting that social networks exhibit small diameter and average path length, it is also important to note that the same is true of many other networks, including routing

[21] Given that twenty to thirty percent is a healthy response rate on a survey, and that having a booklet reach a destination required a chain of subjects to each respond, a twenty five percent rate of reaching the target is impressive, especially in an unpaid experiment.

[22] For example, see Garfield (1979), Kochen (1989), and Dodds, Muhamad, and Watts (2003).

[23] A tree is a network without any cycles, where a cycle is a path consisting of distinct links that start and end at the same node.

[24] For example, see Bollabás (2001) for some theorems bounding diameters in some classes of random graphs.

[25] This stylized fact is captured in the famous "six degrees of separation" of John Gaure's play.

networks, power grids, and networks of neurons.[26] What tends to be a more distinguishing feature of social networks, is their clustering (recall the definition above). Clustering is a simple, but powerful concept that has roots tracing back to work of Simmel (1908), who first explored triads (relationships between triples of individuals). Social networks tend to have significantly higher clustering coefficients than what would emerge if the links were generated by an independent random process. For example, Adamic (1999) finds a clustering coefficient of .11 for a portion of the www, which would compare with an expected clustering coefficient of .0002 for a (Bernoulli) randomly generated network with the same number of links. Figures for other networks are reported in Table 2 below, where we also see relatively high numbers compared to a benchmark random network. For example, if each link is formed with equal probability and independently of each other link, then the probability of two of node i's neighbors being connected to each other is simply the probability with which links are formed. In the first column of Table 2, this would be less than 5/325000, as each node has an average of fewer than 5 links out of a potential number that is more than 325000. The observed clustering of .11 is substantially higher.

2.4.3 Degree Distributions

As discussed in the co-authorship example above, another easily identified property of a social network is its degree distribution. This gives some idea of the variation in the number of links across different nodes, and provides us with some feeling for the shape of a network. Does it have "hub and spoke" like features where there are some very highly connected nodes and others with very few connections, or are connections more evenly distributed? Keeping track of the distribution of degrees in a network can be quite useful. For example, the degrees of the nodes in the Medici marriage network in Figure 1 are 0,1,1,1,1,2,3,3,3,3,3,3,4,4,6. From this we see that the Medici had more than twice the average degree (6 compared to 2.53) and twice the median degree.

One of the early studies documenting degree distributions was by Price (1965) who examined networks of citations among research articles. Price noticed a similar pattern to what we noted in the co-authorship example above, namely that there were more highly connected and lowly connected nodes than what would be expected if links were selected independently and uniformly at random. Much of the recent interest in networks by statistical physicists was sparked by a similar study of Albert, Jeong and Barabasi (1999), which examined the structure of a portion of the www (in the Notre Dame domain). They also found a degree distribution that was distinctly different from what would have been generated by a random process of link formation where all links were equally likely. If links were formed uniformly at random with a link between any two nodes being formed independently of other links

[26] For instance, see Watts (1999) and Newman (2003).

and with a probability p, then the degree distribution would approximate a binomial distribution, and would also be well-approximated by a Poisson distribution (see Section 3.1.1). Again, they found that the degree distribution had "fat tails," in that there were many more nodes with very high and very low degree than would correspond to a binomial or Poisson distribution. In fact, they estimated that the distribution was approximately "scale-free" and followed a "power-law," where the relative frequency of nodes with a degree of k is proportional to $k^{-\gamma}$ for a parameter $\gamma > 1$.[27] The term "power law" clearly refers to the fact that the frequency can be expressed as the degree raised to a power. The term "scale-free" refers to the following property. Consider degree k and some other degree ck, for some scalar c. Their relative frequencies are $k^{-\gamma}/(ck^{-\gamma}$ or c^{γ}. Now consider some other degree k' and another degree ck'. Their relative frequencies are also c^{γ}. Thus, regardless of how we have rescaled things, relative frequencies depend only on relative sizes and not on the absolute scale.

An important caution to the literature is in order here. While it is clear that the degree distributions of many observed networks differ significantly from that of a purely random network; it is not clear that they are "scale-free". This is a point first made by Pennock et al (2002).[28] A standard approach to outlining the degree distribution of many networks has been simply to plot the log(frequency) versus the log(degree) and see whether this "looks" linear. Of course, many things that are far from linear will appear linear on a log-log plot, as most of the data are squeezed into a small portion of the scale on a log-log plot; and such a distribution can be very difficult to distinguish from others, such as a lognormal distribution which can also appear quite linear. Simply fitting a line to the data on a log-log scale does not guarantee that the estimated coefficient means much of anything.

To get a better feeling for the shape of degree distributions, and whether most social networks exhibit features that are close to scale-free, it is possible to consider families of distributions and see which one best fits a given social network. By doing this across different observed social networks, we can get a more precise sense of what the actual degree distributions of the networks are. We can do this with a family of degree distributions that have at one extreme networks whose links are generated uniformly at random, and at the other extreme networks with scale-free distributions. Let us consider a recent study that provides such fits and shows that there are broad differences in the degree distributions of various social networks, and also that some networks that appear to be scale-free in a simple plot are actually better fit by degree distributions that are markedly not scale-free.

[27] Such distributions date to Pareto [?], after whom they are named, and have appeared in a wide variety of settings ranging from income distributions, distribution of city populations, to the usage of words in a language. For an informative overview, see Mitzenmacher [254].

[28] See Eeckhout (2004) for a similar point regarding Zipf's law as applied tocity sizes, and also Ioannides (2004) for a similar point.

Jackson and Rogers (2004) examine a family of degree distributions where the probability that a given node has degree k is given by $P(k) = c(k + rm)^{-(2+r)}$, where c is a constant (ensuring a sum to 1 across k's), m is the average degree, and r is a parameter which varies between 0 and ∞. More specifically, the model is one where new nodes are born over time and can attach to existing nodes either by choosing one uniformly at random or through a search process that makes the likelihood of meeting a given node proportional to the number of links the node already has. r represents the ratio of how many links are formed uniformly at random compared to how many are formed proportionally to the number of links existing nodes already have. As r approaches 0, the distribution converges to be scale-free, while as m tends to infinity the distribution converges to a negative exponential distribution, which corresponds to the degree distribution of a purely uniform and independent link formation process on a network that grows over time.

Using this model, we can back out the relative randomness in the formation process. Fits to a few networks[29] give us an idea of the variation across applications.[30]

Table 2: Comparisons Across Applications

	WWW	Citations	Co-author	Ham Radio	Prison	High School Romance
Number of Nodes	325729	396	81217	44	67	572
Randomness: r	0.5	0.62	3.5	5.0	590	1000
Avg. Degree: m	4.5	5	1.7	3.5	2.7	.84
Avg. Clustering	.11	.07	.16	.06	.001	0

[29] The www data are from an analysis of the links between web pages on the Notre Dame domain of the world wide web from Albert, Jeong, and Barabási [4]. The co-authorship data are from the above cited study by Goyal, van der Leij, and Moraga-González [168]. The citation network consists of the network of citations among all papers that have cited Milgram's (1967) paper or have the phrase "small worlds" in the title, and is from Garfield [153]. The prison data record friendships among inmates in a study by MacRae [241], the ham radio data record interactions between ham radio operators from Killworth and Bernard [215], and the high school romance data collected romantic relationships between high school students over a period of a year and a half in a US high school and is from Bearman, Moody, and Stovel [28]. The number of nodes, average degree, and clustering numbers are as reported by the studies. The estimates on randomness are from Jackson and Rogers (2004). The fits on these estimated r's are remarkably high, with R^2's ranging between 93 and 99 percent.

[30] The clustering figure for the co-author data is actually for total clustering, as the average number is not available but is likely to be higher given that the clustering is decreasing in degree. The clustering for the high school romance network is special because that network is mainly heterosexual in its relationships, and so completed triads do not appear. Even if one looks for larger cycles, there are only five present in the whole network, which would be characteristic of a large network formed at random among two groups.

Thus, we see a marked difference in the degree distributions, as well as clustering and average degree, across different social networks. As these characteristics are the more easily measured features of a social network, and carry a great deal of information about the shape of the network, they are quite useful. As we shall see below, it is also important to note how different the degree distributions are across different social networks, since we can relate the differences in structure to differences in resulting behavior on the networks.

There are many other features of social networks that have been explored, but are beyond our scope here, as some of the basic features discussed above shall already give us a good handle on some of the models that I discuss below.

With a better feeling for what social networks are, and some features that they exhibit, and some basic statistics that help to describe networks, let us now return to the research questions that are a focus of many of the models of networks. As I said in the introduction, my focus here is on two central issues. How networks are formed and how network structure affects the behavior of the individuals involved in the network. I turn to these in order.

3 Modeling Network Formation

As mentioned in the introduction, the models of network formation have come primarily from two sources: the random graph literature (and the subsequent statistical physics literature) and the economics literature (building on game theoretic tools).

Let me emphasize from the outset how different these approaches have been. The random-graph-based literature builds networks either through a purely stochastic process where links appear at random according to some distribution, or else through some algorithm for building links. What this allows one to do, is show how observed networks at some given point in time might have resulted from some stochastic or mechanical process. Although this does not quite answer the *why* behind network formation, it does give us a great deal of insight into the *how*. That is, these sorts of models essentially match observed characteristics back to specific processes. Why one process operates in one setting, and another in a different setting is something essentially beyond the scope of the models.

The economic approach, in contrast, has tended to focus on equilibrium networks,[31] where links are formed at the discretion of self-interested agents who are or control the nodes. A big advantage of this approach is that it naturally incorporates the costs and benefits into the analysis, as the payoffs to agents are part of the model. This enables one to answer questions relating to whether or not the right networks form, in the sense of maximizing the total benefit to society. Such models also give us insight into the *why* behind network formation,

[31] It has also provided some stochastic models of network formation, but mainly as a tool for selection among equilibrium networks. For example, see Jackson and Watts (2002ab) and Goyal and Vega-Redondo (2005).

as they trace network form to the incentives of the agents and the costs and benefits of different links. The main shortcoming of these models is that while they can tell us things about tensions between incentives and efficiency, and trace incentives to primitives relating to costs and benefits, they generally stop short of giving predictions concerning things like which degree distribution should emerge. In a sense, they have not been as well-suited for answering the *how* questions.

I return to more discussion of this in what follows.

3.1 Random Models of Network Formation

3.1.1 Erdös-Rényi (Bernoulli) Random Graphs

The earliest and most extensively studied formal model of network formation is that of purely random graphs, with the canonical example being that of a pure Bernoulli process of link formation. That is, consider a set of nodes and then independently consider each possible link. With probability p have this link be part of the graph, and with probability $1-p$ have that link be absent from the graph. This random graph formation process was explored in detail by Erdös and Rényi (1959, 1960, 1961) and has been studied extensively since then.[32,33]

Figure 3 pictures a network generated through such a procedure, where there are 25 nodes and a probability $p = 1/6$, which will be useful for comparison later on.[34]

There are a number of interesting properties that such Bernoulli networks have.[35] These properties are generally established for large networks; that is, as the number of nodes tends to infinity.

Let us first consider the degree distribution. The probability that any given node i has exactly k links is simply

$$\left(\begin{array}{c} n-1 \\ k \end{array} \right) p^k (1-p)^{n-1-k}. \tag{1}$$

Note that even though links are formed independently, if we want to estimate the fraction of nodes in a network that will have a given degree, there will be some correlation across nodes. For instance, if $n = 2$, then it must be that both nodes have the same degree. As n becomes large, however, the correlation of degree between any two nodes vanishes, as the possible link between them is only one out of the $n-1$ that each might have. Thus, as n

[32] Another closely related random graph model is one where all graphs with n nodes and exactly M edges are considered, and one is randomly selected (where equal probability is placed on each such graph). If $M = np$ and n is large, then many of the properties of the resulting graph are similar to those of the Bernoulli graph process.

[33] See Bollobás (2001) for an extensive overview.

[34] Despite the expected degree of 4, this realized network has an average degree of only 3.28, which puts in the tail of possible networks with respect to average degree.

[35] Again, see Bollobás (2001) for a detailed account of these properties.

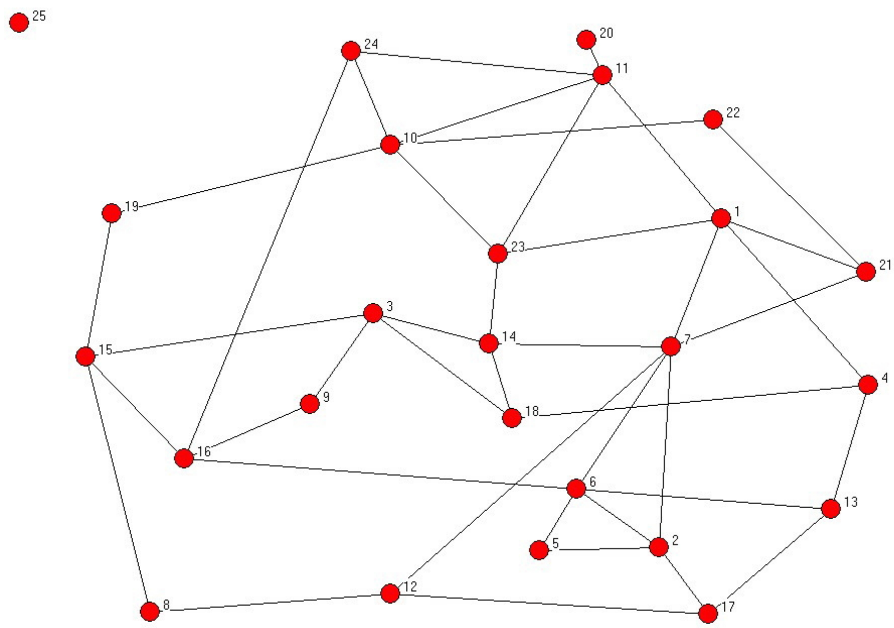


Figure 3: A Bernoulli Random Graph

becomes large, the fraction of nodes that have k links will approach the expression in (1). For large n and small enough p (relative to n), this binomial expression is approximated by a Poisson distribution, so that the fraction of nodes that have k links is approximately[36]

$$\frac{e^{-(n-1)p}((n-1)p)^k}{k!}. \tag{2}$$

This gives us a benchmark degree distribution for comparison.

Interestingly, the random graph generating process exhibits also a number of "phase" transitions as we vary the probability of forming links, p, relative to the number of nodes, n.

When p is small relative to n, so that $p < 1/n$ (average degree is less than one), then the resulting graph consists of a number of disjoint and relatively small components, each of which has a tree-like structure.

Once p is large enough relative to n, so that $p > 1/n$, then we see a "giant component" emerge. That is, almost surely the graph consists of one large component, which contains a nontrivial fraction of the nodes, and all other components are vanishingly small. To get some impression as to the size of the giant component, and why it emerges at the juncture where $p = 1/n$, let us do a simple (heuristic) calculation. Let q be the fraction of nodes in the giant

[36]To see this, simply note that for large n relative to k and small p, $(1-p)^{n-1-k}$ is roughly $(1-p)^{n-1}$, which is approximately $e^{-(n-1)p}$. Then for fixed k and large n, $\left(\begin{array}{c} n-1 \\ k \end{array}\right)$ is roughly $\frac{(n-1)^k}{k!}$.

component. The probability that a node i is not in the giant component is the probability that none of its neighbors are. If node i has degree k_i, then this probability is $(1-q)^{k_i}$. Given the approximation by a Poisson degree distribution, the fraction of nodes outside of the giant component would satisfy the equation

$$1 - q = \sum_k \frac{e^{-(n-1)p}((n-1)p)^k}{k!}(1-q)^k.$$

Since $\sum_k \frac{((n-1)p(1-q))^k}{k!} = e^{(n-1)p(1-q)}$, we end up with an approximation of

$$q = 1 - e^{-q(n-1)p}. \tag{3}$$

There is always a solution of $q = 0$ to this equation. In the case where the average degree is larger than 1 (i.e., $(n-1)p > 1$), and only then, there is also a solution for q that lies between 0 and 1. This corresponds to the phase transition I mentioned above. If average degree exceeds one, then there is a giant component which contains a non-trivial fraction of all nodes, and the size of the giant component is approximately described by the nonzero solution to (3)). For instance, in Figure 3 the giant component contains $q = 24/25 = .96$ of the total nodes. Solving $q = 1 - e^{q(n-1)p}$ when $n - 1 = 24$ and $p = 1/6$ leads to an approximate q of .98.

Why we see just one giant component and all other components are of a much smaller order is fairly intuitive.[37] In order to have two "large" components each having some nontrival fraction of n nodes, there would have to be no links between any node in one of the components and any node in the other. For large n, it becomes increasingly unlikely to have two large components but with absolutely no links between them. Thus, nontrivial components mesh into a giant component, and any other components must be of a much smaller order. Although not an entirely random network (see Table 2), we get an impression of this from the economics co-authorship network of Goyal, van der Leij and Moraga-Gonzalez (2003): it has a total of 81217 nodes and a giant component of 33027 nodes, and yet the second largest component only consists of 30 nodes.

As we continue to increase p, we see another phase transition when p is proportional to $log(n)/n$. This is the threshold at which the network becomes "connected" so that all nodes are path-connected to each other. We again get some impression of why this is happening from our approximation of the size of the largest component in (3). When we plug in $p = log(n)/(n-1)$ then q solves $q = 1 - n^{-q}$, which for large n gives q close to 1. The more formal analysis is quite involved and is detailed in Bollobás (2001). Once we hit the threshold at which the network becomes connected, we also see further changes in the diameter of the network as we continue to increase p relative to n. Below the threshold, the diameter of

[37] For precise bounds on sizes see Bollobás (2001).

giant component is of the order of $log(n)$, then at the threshold of connectedness it hits $log(n)/loglog(n)$, and it continues to shrink as p increases.

One interesting characteristic of Bernoulli random graphs, is that for large n and p that is not too large, we see very little clustering. That is, if two links, say ij and jk, are both present and then ask with what probability ik is present, the answer is simply p. So, if p is relatively small, then so is clustering. In particular, if we examine large social networks, where p should be close to 0, then the clustering of a network goes to 0. Having p be close to 0 when n is large would be necessary if, for instance, there is some upper bound on the average degree. This is something we expect in many social networks as there is some bound on the number of links an individual can maintain. In fact, if p is small enough and n is large enough, it is not only that we expect low clustering, but in fact we do not expect any loops or cycles in the network; that is, with high probability all components of the network will be trees (see Bollobas (2001) for details).

While this is just a quick look at some of what is known about Bernoulli (Erdös-Rényi) random graphs, it gives us some feeling for some properties of purely random networks. This is useful since such Bernoulli random graphs provide a relatively good match for some observed networks (e.g., the prison friendships and high school romance networks[38] reported in Table 2), and also because some of the types of phase transitions and features observed in these networks are also observed in other random network models. However, although the Bernoulli random graphs studied by Erdös and Rényi and others provide a useful benchmark model for social networks and fit in a few cases, their lack of clustering, among other things means that they lack some basic features exhibited by many observed social networks. This has led researchers to explore other sorts of random network models.

3.1.2 Markov Graphs and p* networks

There are various generalizations of Bernoulli random graphs that have been useful in statistical analysis of observed networks. In particular, Frank and Strauss (1986) identified a class of random graphs that generalize Bernoulli random graphs, which they called "Markov graphs". Such random graph models were later introduced to the social networks literature by Wasserman and Pattison (1996) under the name of p* networks, and further studied and extended in various directions.[39] The basic idea is to allow for specific dependencies in a network.

[38] Again, the high school romance network is mainly bipartite in nature and so would be best modeled through a modified random model where links only form across gender, but still exhibits remarkable randomness in the structuring of its links.

[39] For instance, see Pattison and Wasserman (1999) for an extension to multiple interdependent networks on a common set of nodes.

To get some feeling for the importance of introducing dependencies, let us consider the clustering discussed above. As just discussed, with a relatively low probability of any given link being present, if all links are independently determined, then the clustering ratio will tend to be low, and too low to match many observed networks. However, suppose that one allows for conditional dependencies, so that the conditional probability of a link ik depends on whether ij and jk are present. In general, it can be quite difficult to see how to introduce such dependencies, as they will tend to interact with each other which could make it impossible to specify the probability of different graphs in a tractable manner. For instance, if the conditional probability of a link ik depends on whether ij and jk are present, but also on any other adjacent pairs being present, and the conditional probability of jk depends on other adjacent pairs being present, etc.; we end up with a complicated set of dependencies. The important contribution of Frank and Strauss (1986) is to make use of a theorem by Hammersley and Clifford (see Besag (1974)) to derive a simple log-linear expression for the probability of any given network in the presence of arbitrary dependencies.

One particularly useful result of Frank and Strauss (1986) can be expressed as follows. Consider a random network on n nodes. Let us keep track of the dependencies between links by another graph, D, that is a graph among all of the $n(n-1)/2$ possible links (or $n(n-1)$ possible directed links if the network is directed). So, D is not a graph on the original nodes, but a graph whose nodes are all the possible links. The idea is that if ij and jk are connected in D, then there is some sort of conditional dependency between them, possibly in combination with other links. Thus, D captures which links are dependent on which others, possibly in quite complicated combinations. For example, the Bernoulli model is one where D is empty, as all links are independent. If instead, we wish to capture the idea that there might be clustering, then we would like the link ik to depend on the presence of ij and kj for each possible j. Thus, D would have ik connected to each other link that contains either i or k.

Let $C(D)$ be all the cliques of D; that is, all of the completely connected subgraphs of D (where the singleton nodes are considered connected subgraphs). So, in the case of a Bernoulli random graph $C(D)$ would simply be the set of all links ij. In the case of the clustering dependence just mentioned above, the set $C(D)$ would include all individual links and also all of the triads (sets of the form $\{ij, jk, ik\}$). Given a generic element $A \in C(D)$, let $I_A(g) = 1$ if $A \subset g$ (viewing g as a set of links), and $I_A(g) = 0$ otherwise. So, if A is a triad $\{ij, jk, ik\}$, then $I_A(g) = 1$ if each of the links ij, jk and ik are in g, and $I_A(g) = 0$ otherwise. Then, Frank and Strauss show that Hammersley and Clifford's theorem implies that the probability of a given network g depends only on which cliques of D that it contains, and that it can be written as

$$log(\Pr[g]) = \sum_{A \in C(D)} \alpha_A I_A(g) - c, \tag{4}$$

where c is a normalizing constant, and the α_A's are other free parameters.

In general, given that D can be very rich and that the α_A's can be chosen at will, this allows for an almost arbitrary probability specification. The difficulty, and art, in applying this type of model in practice is in specifying the dependencies sparingly and some restrictions on the α_A's so that the resulting probabilities are simple and practical. For some certain kinds of dependencies, the expressions can be quite simple and useful (e.g., see Anderson, Wasserman and Crouch (1998)).

To see how the expressions can simplify, let us consider the clustering dependency we mentioned above. This means that $C(D)$ is just the set of all links and all triads (triplets of the form $\{ij, jk, ik\}$). To simplify things further, let us also suppose that there is a symmetry among nodes, so that the probability of any two networks that have the same architecture but possibly different labels on the nodes is identical. This means that the α_A's are the same across all A's that correspond to single links, and the same across all A's that correspond to triads. Thus, the expression in 4 simplifies substantially. Let $n_1(g)$ be the total number of links in g, and let $n_3(g)$ be the total number of completed triads in g. Then there exist α_1, α_3 and c such that (4) becomes

$$log(\Pr[g]) = \alpha_1 n_1(g) + \alpha_3 n_3(g) - c.$$

This then provides us with a simple generalization of Bernoulli random graphs (which are the special case where $\alpha_3 = 0$), which will allow us to control the frequency of clusters. That is, we can adjust the parameters so that graphs that have more substantial clustering will be relatively more likely than graphs that have less clustering (for instance, by increasing α_3).[40]

While such a model can be cumbersome as we try to capture more complicated dependencies, it still provides a powerful statistical tool for testing for the presence of some specific dependency. One can test for significant differences between fits of a model where such dependencies are present and a model where such dependencies are absent.[41]

3.1.3 Rewired Lattices and Clustering

Watts and Strogatz (1998) looked at another variation on a Bernoulli network, with a particular question in mind. They wanted to generate networks that exhibit both relatively low diameter and nondegenerate clustering. They developed a model that mixes purely random

[40] See Park and Newman (2004) for some derivations of clustering probabilities for this example.

[41] Obviously, the validity of the test depends on the appropriateness of the rest of the specification of the model which can be a problem in practice.

link formation with a lattice structure. The structure of their model is easy to understand. I will discuss a particular example, and refer the reader to Watts (1999) for more detailed derivations and simulations of the model.[42]

Suppose we start with a very structured network that exhibits a high degree of clustering. For instance, let us construct a large circle, but then connect a given node to the nearest four nodes rather than just its nearest two neighbors.

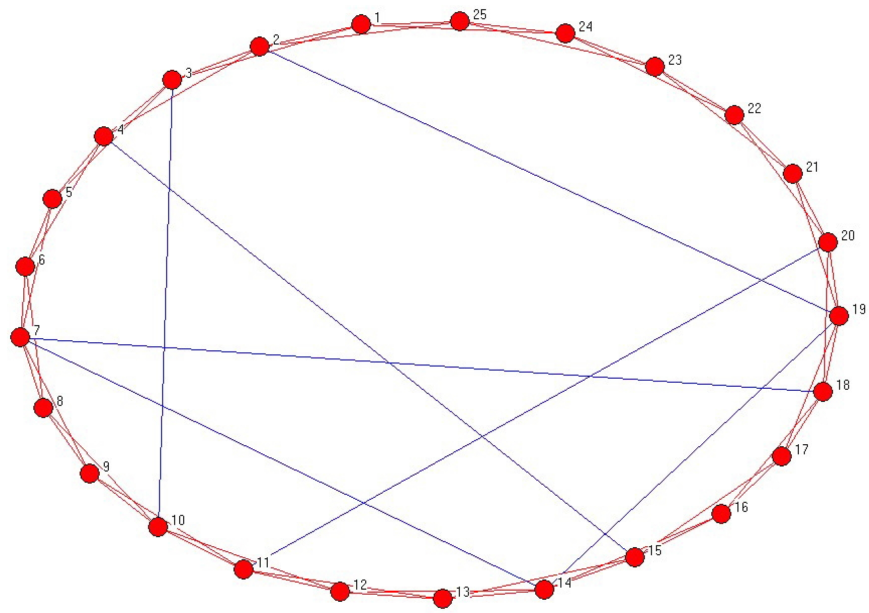


Figure 4: A Ring Lattice with Randomly Added Links

In such a network, each node's individual clustering coefficient will be 1/2. To see this, consider some set of consecutive nodes 1, 2, 3, 4, 5, that are part of such a network for a large n. Consider node 3, which is connected to each of the other nodes. Out of all the pairs of 3's neighbors $(\{1,2\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{4,5\})$, we see that half of them are connected $(\{1,2\},\{2,4\},\{4,5\})$

Note, however, that the diameter of such a network is on the order of $n/4$, which is out of line with what we observed in Table 1 and Section 2.4.1, where the diameter was on the order of 20 for networks with hundreds of thousands of nodes. The main point of Watts and Strogatz (1998) is that by starting with such a highly clustered ring lattice, and then randomly rewiring enough (but not too many) links, we can end up with a network that has a much smaller diameter but still has substantial clustering. The rewiring can be done by randomly selecting some link ij and disconnecting it and then randomly connecting i to

[42]See also Barrat and Weight (2000).

another node k chosen uniformly at random from those whom i is not already connected to. Of course, eventually if too much rewiring is done, the clustering will vanish.[43] The interesting region is where enough rewiring has been done to substantially reduce (average and maximal) path length, but not so much that clustering vanishes. The key to this process is that the short cuts introduced by relatively few rewirings can dramatically decrease path lengths, and that this is a nonlinear relationship.

A slight variation of this original model proposed by Newman and Watts (1999) (see also Monasson (1999)) is easier to analyze in terms of its properties. That model starts with the same sort of ring lattice structure, but instead of rewiring links randomly, it simply adds links randomly. While precise diameter and path length numbers have still not been obtained for this model, we can easily derive upper bounds based on those for the corresponding random graphs, and in terms of diameter these bounds should not be too far off (at least when the corresponding random graph, ignoring the ring lattice, would be connected by itself). For instance if we introduce more than $log(n)/n$ random links, then we can expect a diameter of no more than $log(n)/loglog(n)$ (given what is known about Bernoulli random graphs).

While the rewiring of a ring lattice provides the high clustering and low path lengths that we observe in many social networks, the resulting degree distributions are far from what is observed. In particular, in order to keep a reasonably high clustering coefficient in such a model, the initial ring lattice structure has to stay largely intact and has to represent a non-trivial fraction of an average nodes links. This means that the resulting degree distribution has a great deal more regularity and less variance than what is generally observed.

3.1.4 Preferential Attachment and Scale-Free Degree Distributions

As mentioned above, in order to match the degree distributions that are observed in many social networks, one needs a process of link formation that differs from the pure Bernoulli (Erdös-Rényi) process, as observed distributions often exhibit fatter tails.

The ideas behind generating distributions with such "fat tails" date to Pareto (1896), for which the standard power distribution is named, and continued in Yule (1925) and were really crystallized and formalized by Simon (1955). The underlying principle is what is often referred to as a "rich-get-richer" structure, or essentially something akin to a lognormal growth system. If objects grow in size at a rate proportional to their current size, then we should expect "fat tails" in the distribution of sizes. In particular, Simon pointed out that in a system where objects are born at different times, and then grow lognormally once they are born, the resulting distribution of object size will follow a power-law or scale-free

[43] That is, the network will take on the features of a random network, which has vanishing clustering as the number of nodes becomes large and the number of links per node is not growing too rapidly.

distribution.[44] This can be applied to distributions of wealth and city sizes, among many other things.[45]

Price (1965) first observed that some networks (in particular, citation networks) had degree distributions with special features. In a later seminal paper, Price (1976) adapted Simon's (1955) ideas to the setting of a growing (citation) network in order to generate scale-free degree distributions. The idea was that the number of citations that papers would gain over time were proportional to the number of citations they already had.[46] In the recent literature, such a process has been referred to by the name of "preferential attachment," as coined by Barabasi and Albert (2001), who developed a model similar to Price's (1976) model except it is undirected, while Price's was directed.[47]

Let me briefly describe such a model, as it is useful in illustrating some of the techniques from that literature. Consider a system where a new node is born at each date. So let us index nodes by their date of birth $i \in \{0, 1, 2..., t, ...\}$. Upon birth (and only then), each new node forms m links with pre-existing nodes.[48] The new node selects the nodes to link to in a random manner, but with a probability that is proportional to the number of links that each given node already has. For example, if an existing node i has twice as many links as some other node j, then it is twice as likely to get a given link from the newborn node. So, roughly, the probability that any given existing node i gets a new link at time t is m times its degree relative to the overall degree of all existing nodes that time t, or $m\frac{k_i(t)}{\sum_{j=1}^{t} k_j(t)}$, where $k_i(t)$ is node i's degree at time t and $\sum_{j=1}^{t} k_j(t)$ is the normalization by the total degree of all nodes. As there are tm total links in the system at time t, $\sum_{j=1}^{t} k_j(t) = 2tm$. Thus, the

[44] Another explanation behind power laws is the idea of "HOT" (highly optimized tolerance) systems that underlies Carlson and Doyle [76] and Fabrikant, Koutsoupias, and Papadimitriou [123]. That important idea addresses systems that are centrally optimized, rather than self-organizing.

[45] See Mitzenmacher (2004) for a nice overview.

[46] One can explain such a system via a simple process. If researchers randomly find a paper (which these days can be explained via a key-word search) and then search for additional papers via the references they find in the first paper, then the chance of being found is roughly proportional to the number of citations that a paper already has.

[47] See Newman (2003) (and also Mitzenmacher (2004)) for more discussion of the various naming of such processes and their development.

[48] There are some details to worry about in starting such a process. Early nodes may not end up being able to form m links. If we count early nodes as having m links, regardless of how many they actually formed, then the process is well-defined. Alternatively, one can start with some initial connected network of at least m nodes already in place. Also, a node can form at most one link to any given other node. So, one can either follow a convention of allowing a node to try to form more than one link with an existing node and then treating more than one link between two nodes as just one link (or instead one could re-randomize whenever a node tries to form a link with a node that it has already linked to). The network in Figure 5 was formed by treating nodes 1 and 2 as if they had degree 2, and by allowing nodes to form multiple links to a single node, but then counting them as just one link (for instance, note that node 3 only formed a link with node 2 and not node 1).

probability that any given existing node i gets a new link in period t is $\frac{k_i(t)}{2t}$.

This results in a well-defined stochastic process (an infinite Markov chain). As such the steady state distribution of a Markov process can be hard to solve for explicitly, the system is often approximated. For instance, if we approximate the random discrete time system by a continuous time system, where the degree of each node grows deterministically at the expected rate, then we can solve it explicitly. This is termed a "mean-field" approximation.

So, in this system, given that a node i is expected to gain roughly $\frac{k_i(t)}{2t}$ links in period t, the mean field approximation is to solve the system where

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t}.$$

This differential equation with initial condition $k_i(i) = m$ leads to a solution of

$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}.$$

Thus, nodes are born over time and then grow. The system is now much simpler than the random system, in that the degrees of nodes can be ordered by their ages. The oldest nodes are the largest. To find out what the fraction of nodes is that exceeds some given level k at some time t, we just need to identify which node is at exactly level k at time t, and then we know that all nodes born before then are the nodes that are larger. Let $i_t(k)$ be the node which has degree k at time t, or such that $k_{i_t(k)}(t) = k$. From our above equation, we know that

$$\frac{i_t(k)}{t} = \left(\frac{m}{k}\right)^2.$$

The fraction of nodes that have degree smaller than k at time t are then the proportion born after node $i_t(k)$, or born after time $t\left(\frac{m}{k}\right)^2$. Thus, the distribution function is

$$F_t(k) = 1 - m^2k^{-2}.$$

This has a corresponding density or frequency of[49]

$$f(k) = 2m^2k^{-3}.$$

Thus, we obtain a scale-free distribution with an exponent of -3.[50]

To get some feeling for such a network, and how it might differ from the previous random graph models, consider the following figure of a 25 node network which was generated using such a preferential-attachment process where each new node forms two links.[51]

[49] Note that the expression for F_t is in fact independent of t (which is an artifact of the continuous time mean-field approximation). Thus, the subscript is dropped from the expression.

[50] The specifics of the exponent -3, comes from the -2 in the distribution function, which is traced back to the fact that each link is shared by a new and old node. If these were in different proportions, the exponent would change. See Simon (1955) for more discussion of this, and Jackson and Rogers (2004) for an alternative model with additional variation in the exponent.

[51] Node 1 formed no links at birth. Node 2 formed only a link to 1. And then the process was well defined

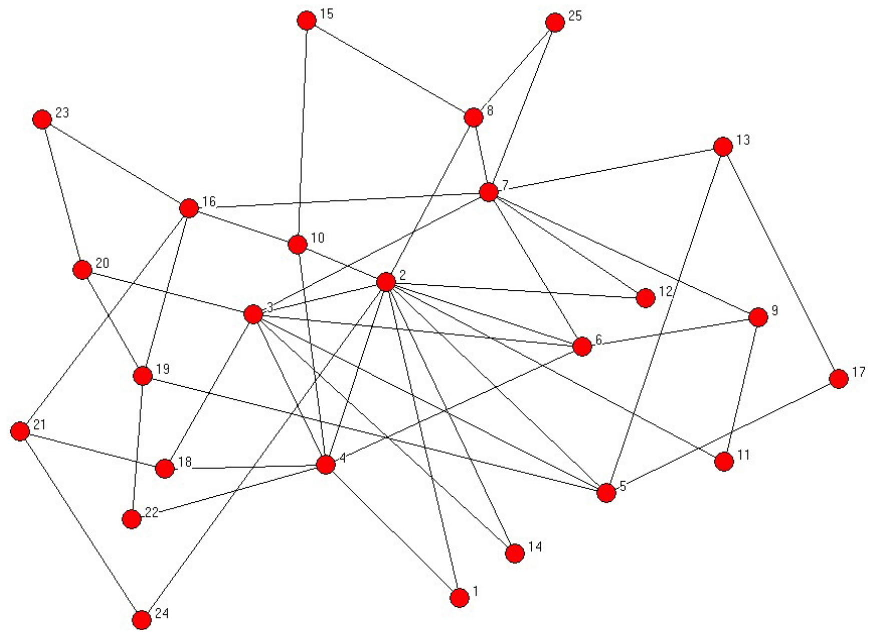


Figure 5: A Network Formed by Preferential Attachment

The nodes are indexed by their birth dates, and we see that the older nodes tend to have much higher degrees. For instance, node 2 has degree 11 while nodes 22 to 25 have only degree 2, or the links they form at birth.[52] This network looks very different from the earlier models which had approximately the same average degree (Figures 3 and 4).

3.1.5 Hybrid Models

From the models we have discussed so far, we see that in each case there is some deficiency. The purely random graphs analyzed by Erdös and Rényi do not exhibit the clustering or degree distributions that match many observed networks. The rewired ring lattices of Watts and Strogatz (1998) do not exhibit degree distributions matching observed networks. Preferential attachment generates scale-free degree distributions that help account for the fat-tailed degree distributions observed in many applications. However, it turns out that preferential attachment generates networks that do not exhibit any clustering. Moreover, as we saw from Table 2, degree distributions differ substantially across applications and tend to lie somewhere

from then on.

[52]It is clear that many networks have degree distributions that are not so purely age dependent. By adding a weighting parameter or "fitness," it is easy to extend the model so that some younger nodes can overtake older nodes because they are more attractive to link to (see Bianconi and Barabasi (2001)). In such a model each node is born with a level of attractiveness, and the probability that they are attached to is proportional to this level of attractiveness times the number of links they have, properly normalized.

between purely random and purely scale-free.

Thus, in order to match observed networks, we need several characteristics: relatively low diameter, nontrivial clustering, and a degree distribution that spans between purely random and scale-free networks. Recent models have made progress in generating networks that are closer to observed networks. Pennock et al (2002), show that by use of a mixed model where some links are formed uniformly at random and others are formed via preferential attachment results in a degree distribution that spans between random and scale-free degree distributions. There are other models that are hybrids of random and preferential attachment (e.g., Kleinberg et al (1999), Kumar et al (2000) , Dorogovtsev and Mendes (2001), Levene et al (2002), and Cooper and Frieze (2003)). Interestingly, most of these ignore the fact that the resulting degree distributions are not scale-free, but instead try to show that the distribution is at least approximately scale-free for large degrees. Pennock et al (2002) were the first to recognize the fact that many observed networks were not really scale-free, and thus that a hybrid model could better match observed degree distributions. Unfortunately, that model does not provide any clustering.

In order to generate clustering, and still have some sort of scale-free aspect to a degree distribution, Klemm and Eguíluz (2002, 2002b) have a variation of the preferential attachment model where nodes are declared either active or inactive. A new node enters as "active" and then some existing active node is randomly de-activated (with a probability inversely proportional to its degree). New nodes attach to each active node. Then with a probability μ, each of these links is rewired to a random node in the population chosen according to preferential attachment. This process thus has a fixed number of "active" nodes, and the fact that each entering node ends up hooked up to a proportion $1 - \mu$ of them, and that the list of active nodes only changes by one each period, results in significant clustering. The preferential attachment structure results in the scale-free distribution and small diameter. However, this sort of model only generates scale-free distributions.

Jackson and Rogers (2004) show that a hybrid model can result in all of the features of high clustering, small diameter, and a degree distribution that spans between purely random to scale-free. In that model, nodes are born over time and each node forms m links, just as in the preferential attachment model described above. However, instead of forming links randomly, the new nodes meet (and form links to) some existing nodes purely at random and then also meet some neighbors of these nodes.[53] Meeting neighbors introduces an element of preferential attachment, as the chance of meeting a node in that manner is proportional to how many neighbors it has. As the ratio of how many nodes are met at random compared to how many of their neighbors are met is varied, this process spans between one of completely

[53] See Vazquez (2003) for a related process which also exhibits clustering, where links are formed by first entering at a randomly selected node and then following a path emanating from the node.

random link formation to one of pure preferential attachment. However, it exhibits nontrivial clustering between these extremes, since then a new nodes will often form a links both to an existing node and one of its neighbors, thus forming a completed triangle. Jackson and Rogers show that such a model can fit observed networks well on several dimensions at once (see Table 2).

3.2 Strategic Models of Network Formation

As I have provided extensive discussion of strategic models of network formation elsewhere (Jackson (2003, 2004)), here I will present a few examples to illustrate some key points about the literature on network formation that has emerged from economics and game theory.

There are two key aspects of an economic/game theoretic approach to modeling network formation:

(i) agents derive some utility from the network, and thus there is an overall societal welfare corresponding to any network that might arise, and

(ii) links are formed at the discretion of the agents who are (or control) the nodes, and resulting networks can be predicted through notions of equilibrium or possibly stochastic dynamic processes.

While economists are so used to looking at costs and benefits and using utility based models that (i) would be taken for granted; it is important to note that this perspective on network analysis is a key distinguishing feature from the "random" models discussed above. This is an important feature that allows one to assess the implications of various networks or formation processes, to deduce whether "good" networks are emerging from society's perspective. Having utilities assigned to networks is, of course, also a prerequisite for an equilibrium analysis, (ii), which complements the "random" processes and provides different insights into network formation. Being able to evaluate the consequences of various network structures is necessary in order to move the study of social networks beyond a purely descriptive exercise, and having welfare measures and outcomes associated with different networks is essential in this regard. Also when integrated with an equilibrium analysis, we can analyze and understand the potential conflict that arises between the networks emerging through the choices of the parties involved and the networks that are best from a societal perspective.

Another important point to emphasize, is that such a game theoretic perspective can help answer the questions as to *why* certain network features might appear. I will discuss this a bit more below, with respect to contrasting views of "small-worlds" phenomena. I will also discuss the limitations of an economic/game theoretic approach, and the potential for hybrid approaches, which combine some randomness and heterogeneity with economic motivations for link formation.

3.2.1 An Economic Approach

Some of the first models bringing explicit utilities and choice to the formation of social links, were in the context of modeling the tradeoffs between "strong" and "weak" ties in labor contact networks. These models were by Boorman (1975) and Montgomery (1991), and explored findings and hypotheses about different strengths of ties and their role in finding employment that were originally due to influential work by Granovetter (1974). Granovetter had observed that when individuals obtained jobs through their social contacts, while they sometimes did so through strong ties (people whom they knew well and interacted with on a frequent basis), they also quite often obtained jobs through weak ties (acquaintances whom they knew less well and/or interacted with relatively infrequently). This led Granovetter to coin the phrase "the strength of weak ties." There have been many studies that have followed on Granovetter's work, and many different explanations for such phenomena, as well as more explicit distinctions between what constitutes a strong versus a weak tie in various settings. Boorman's article and Montgomery's articles provided explicit models where costs and benefits could be assigned to strong and weak ties, and tradeoffs between them could be explored.

From a completely separate perspective, another use of utility functions in a network context emerged in the work of Myerson (1977). Myerson was originally interested in characterizing a cooperative game theoretic solution concept, the Shapley value, without directly imposing an additivity axiom. This led him to analyze a class of cooperative games[54] that were augmented with a graph structure. In particular, in these games the only coalitions that could produce value are those that are pathwise connected within the underlying graph. He thought of the graphs as indicating the possible cooperation or communication structures. So, starting with some given cooperative game and then augmenting it with such a graph, one ends up with a new cooperative game where the worth of any coalition is determined by how it is partitioned by the graph.[55] In that framework, a natural analog of the Shapley

[54] For those not familiar with cooperative games, a standard formulation specifies a value or worth for every possible set of players. A solution, such as the Shapley Value, then predicts or suggests how the total value of the society as a whole (the "grand coalition"), should be split between its members and how that depends on the values that are generated by all the possible subcoalitions. The reason that the information of the values of subcoalitions is important, is that it provides information about how much different players contribute to society as one can calculate, for instance, how much value would be lost if a given player were removed from the society or from some given subcoalition.

[55] For example, considering a coalition 1,2,3 and a graph that just has a link between 1 and 2, means that the coalition would generate value as if it were just 1,2 instead of 1,2,3 (normalizing isolated players to have value of 0).

value (now termed "The Myerson Value") emerges and can be characterized with some simple axioms. Aumann and Myerson (1988) then recognized that different graph structures led to different allocations of value to the agents in the society, and so they studied a specific extensive form game where links are considered one-by-one according to some exogenous order, and formed if both agents involved agree. While that game turns out to be hard to analyze even in three-person examples, it was an important precursor to the more recent economic literature on network formation.[56]

In contrast to the cooperative game setting, Jackson and Wolinsky (1996) explicitly considered networks, rather than coalitions, as the primitive. Thus rather than deducing utilities indirectly through a cooperative game on a graph, they posited that networks were the primitive structure and agents derived utilities based on the network structure in place. Once we have utility being derived from networks, we can take a game theoretic approach to modeling network formation by modeling the formation of links via the decisions of self-interested maximizing players.

As with any game theoretic setting, there are different approaches to modeling equilibrium. A standard equilibrium concept such as Nash equilibrium is not well suited to modeling network formation, as the consent of two players is generally needed to form a link or relationship.[57] For example, if we simply consider a game where each agent announces the links he wishes to form and we form links that are jointly announced, it is always a Nash equilibrium to have no links form. Each player announces an empty set of links since he or she (correctly) anticipates that all other players will do the same.

There are various ways around this, and a very simple one is to define a simple stability notion directly on networks. This was the approach followed by Jackson and Wolinsky (1996) who defined the following notion of pairwise stability. A network is pairwise stable if no player wants to sever a link and no two players both want to add a link.

More formally, let $u_i(g)$ denote the net utility that agent i receives under the network g, inclusive of all costs and benefits.

A network g is *pairwise stable* if

(i) for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$, and

(ii) for all $ij \notin g$, if $u_i(g + ij) > u_i(g)$ then $u_j(g + ij) < u_j(g)$.

[56] See Slikker and van den Nouweland (2001) for an overview of much of the cooperative game theoretic literature that has followed on communication and cooperation structures based on networks.

[57] There are some exceptions. In some purely directed settings, it is possible to form a link without the destination node's consent - such as forming a link to a web page or citing a paper. In such cases, the issue of mutual consent does not arise and a solutions such as Nash equilibrium and its refinements can be used. See Jackson (2004) for more discussion and references on such equilibrium modeling issues.

The requirement that no player wishes to delete a link that he or she is involved in implies that a player has the discretion to unilaterally terminate relationships that he or she is involved in. The second part of the definition can be stated in various ways. In order for a network to be pairwise stable, it is required that if some link is not in the network and one of the involved players would benefit from adding it, then it must be that the other player would suffer from the addition of the link. Another way to state this is that if we are at a network g where the creation of some link would benefit both players involved (with at least one of them strictly benefiting), then the network g is not stable, as it will be in the players' interests to add the link.

While pairwise stability is natural and quite easy to work with, there are limitations of the concept. For instance, it only considers deviations one link at a time, and by at most two players at a time. This is a current area of research (see Section 5), but nonetheless pairwise stability serves as a sensible starting point.

Given that we have well-defined payoffs to players as a function of the network, there are two obvious and standard notions of welfare that we can apply. The first is simply a utilitarian principle, which is to say the "best" network is the one which maximizes the total utility of the society. This notion was referred to as "strong efficiency" by Jackson and Wolinsky (1996), but I will simply refer to it as efficiency.

Let $v(g) = \sum_i u_i(g)$ be the total value that accrues to society as a function of a network g. A network g is *efficient* relative to v if $v(g) \geq v(g')$ for all $g' \in G(N)$.

It is clear that there will always exist at least one efficient network, given that there are only a finite set of networks.

The other natural notion of efficiency is that of Pareto efficiency. A network g is *Pareto efficient* relative to $(u_1, \ldots, u_n)$ if there does not exist any $g' \in G$ such that $u_i(g') \geq u_i(g)$ for all i with strict inequality for some i.

To understand the relationship between the two definitions, note that if g is efficient relative to $v = \sum_i u_i$ then it is clearly Pareto efficient relative to $(u_1, \ldots, u_n)$. The converse is not true. What is true is that g is efficient relative to v if and only if is Pareto efficient relative to all $(\widehat{u}_1, \ldots, \widehat{u}_n)$ such that $\sum_i \widehat{u}_i = v$.

Thus, efficiency is a stronger notion and is the more natural notion in situations where there is some freedom to reallocate value through transfers. Pareto efficiency is a less decisive notion, often admitting many networks, but it might be more reasonable in contexts where the payoff functions are fixed, and no transfers are possible.

3.2.2 The Connections Model

A simple model of social connections, from Jackson and Wolinsky (1996), is useful for illustrating the relationship between efficient and pairwise stable networks.

In the connections model, links represent social relationships between players (nodes); for instance friendships. These relationships offer benefits in terms of favors, information, etc., and also involve some costs. Moreover, players also benefit from indirect relationships. A "friend of a friend" also results in some indirect benefits, although of a lesser value than the direct benefits that come from a "friend." The same is true of "friends of a friend of a friend," and so forth. The benefit deteriorates in the "distance" of the relationship. This is represented by a factor δ_{ij} that lies between 0 and 1, which indicates the benefit from a direct relationship between i and j and is raised to higher powers for more distant relationships. For instance, in the network where player 1 is linked to 2, 2 is linked to 3, and 3 is linked to 4; player 1 gets a benefit of δ_{12} from the direct connection with player 2, an indirect benefit of $(\delta_{13})^2$ from the indirect connection with player 3, and an indirect benefit of $(\delta_{14})^3$ from the indirect connection with player 4. For $\delta_{ij} < 1$ this leads to a lower benefit from an indirect connection than a direct one. Players only pay costs, however, for maintaining their direct relationships.

We can write the net utility or payoff $u_i(g)$ that player i receives from a network g as

$$u_i(g) = \sum_{j \neq i:\ i \text{ and } j \text{ are path-connected in } g} (\delta_{ij})^{p_{ij}(g)} - \sum_{j \neq i:\ ij \in g} c_{ij},$$

where $p_{ij}(g)$ is the number of links in the shortest path between i and j and $c_{ij} > 0$ is the cost for player i of maintaining a link with j.

To see how this works, let us consider the special case, termed the "symmetric connections model," where the cost and benefit parameters are identical for all agents, so there exist $1 \geq \delta \geq 0$ and $c \geq 0$ such that $\delta_{ij} = \delta$ and $c_{ij} = c$ for all ij.

Then, for instance, we can easily deduce utilities in the network pictured in Figure 6.

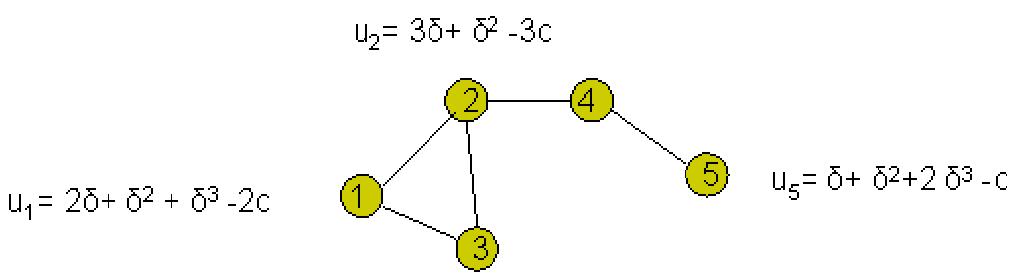


Figure 6: An Example of Payoffs in the (Symmetric) Connections Model

The highly stylized nature of the connections model allows us to begin to answer questions regarding which networks are efficient, or "best" from society's point of view, as well as which networks are likely to form when self-interested players choose their own links as modeled through pairwise stability.

Let us examine the efficient networks. They are characterized as follows in the symmetric connections model:

(i) the complete network if $c < \delta - \delta^2$,

(ii) a star encompassing all nodes if $\delta - \delta^2 < c < \delta + \frac{(n-2)}{2}\delta^2$, and

(iii) the empty network if $\delta + \frac{(n-2)}{2}\delta^2 < c$.

The intuition behind this is very clear. If costs are very low, (i), it will be efficient to include all links in the network. In particular, if $c < \delta - \delta^2$, then adding a link between any two agents i and j will always increase total welfare. This follows because they are each getting at most δ^2 of value from any indirect connection between them, and since $\delta^2 < \delta - c$ the value of a direct connection between them increases their utilities (and might also increase the utilities of other agents). When the cost rises above this level, so that $c > \delta - \delta^2$ but c is not too high, it turns out that the unique efficient network structure is to have all players arranged in a "star" network. This can be seen from several observations, and a careful proof is not much more complicated. The first observation is that a star network involves the minimal number of links $(n-1)$ needed to connect all individuals. The second is that in a star network all nodes are within at most two links from one another. The third observation is that when $c > \delta - \delta^2$, then a path of length two between two nodes generates more utility than a path of length one. A star has the minimal number of links, and has all nodes at distances of two or less, and the most possible at a distance of two out of all networks that connect all individuals. It is also easy to check by direct calculations, that if a small star generates positive total utility then a larger star generates more, and that a single star outperforms separate stars. Thus, if it is efficient to connect agents at all when $c > \delta - \delta^2$, then we should do it through a single star. The calculation in (iii) comes from checking whether or not the utility of a star including all nodes is positive.

Thus, in the connections model the set of efficient networks have a remarkably simple characterization: either costs are so low that it makes sense to add all links, or are so high that no links make sense, or costs are in a middle range and the unique efficient architecture is a star network.[58]

We can now compare the efficient networks with those that arise if agents form links in a self-interested manner. The pairwise stable networks are as follows.

(i) If $c < \delta - \delta^2$, then the complete network is the unique pairwise stable network.

[58] This characterization of efficient networks actually holds for a much broader set of environments. Essentially in any situation where utility depends on minimal distances between nodes and there is some sort of decay of value with distance and there are symmetries across agents, the same conclusions hold (see Bloch and Jackson (2003)).

(ii) If $\delta - \delta^2 < c < \delta$, then a star encompassing all nodes is pairwise stable, and there are also other pairwise stable networks.

(iii) If $\delta < c < \delta + \frac{(n-2)}{2}\delta^2$, then all pairwise stable networks are inefficient, and are such that each agent has either no links or at least two links.

(iv) If $\delta + \frac{(n-2)}{2}\delta^2 < c$, then the empty network is the unique pairwise stable network.

In the case where costs are very low $c < \delta - \delta^2$, the direct benefit to the agents from adding or maintaining a link is positive, even if they are already indirectly connected. Thus, in that case the unique pairwise stable network will be the efficient or complete network. When costs are very high, then no links form and again we have an efficient outcome. The more interesting cases in the middle ranges of (ii) and (iii), so that the star is the efficient network, but is only sometimes pairwise stable and even then not uniquely so. It is easy to see why if $c > \delta$, then the efficient (star) network will not be pairwise stable. This follows since the center player gets only a marginal benefit of $\delta - c < 0$ from any of the links. This tells us that in this cost range there cannot exist any pairwise stable networks where there is some player who just has one link, as the other player involved in that link would benefit by severing it. For various values of $c > \delta$ there will exist nonempty pairwise stable networks, but they will not be star networks: as just argued, they must be such that each connected player has at least two links.

Parts (ii) and (iii) of the above description of pairwise stable networks are illustrated in the following figure.

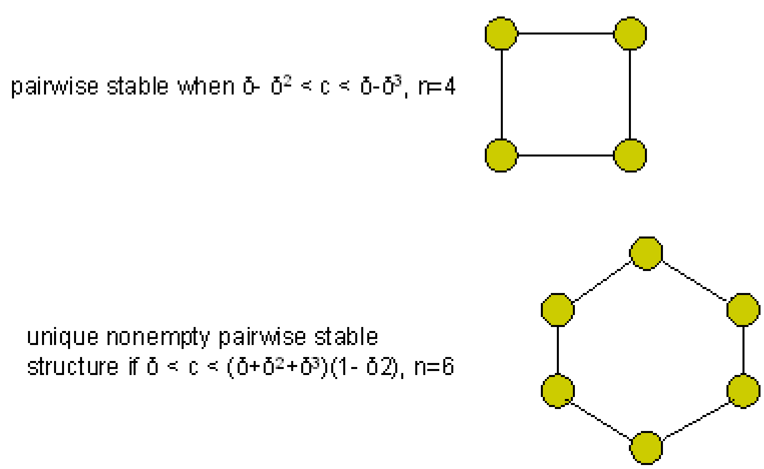


Figure 7: Two Pairwise Stable Networks in the Symmetric Connections Model

This simple model makes it obvious that there will be situations where individual incentives are not aligned with overall societal benefits. While this connections model is highly stylized, it still captures some basic insights about the payoffs from networked relationships and it shows that we can begin to understand the incentives that underlie network formation and see when resulting networks are efficient.

This model also raises some interesting questions, that we can examine further. For example, the central player in a star is one that we might intuitively think of as occupying a particularly powerful and central position, and in many contexts we would actually expect that player to be better off than other players due to the centrality. However, in the context of the connections model it can be that the central player is actually the worst off as he or she bears the greatest costs. An important missing ingredient is that there is no bargaining or transfer of favors or payments that might compensate the center player. If we allow other players to offer to "pay" the central player for maintaining ties, this can dramatically change what turns out to be stable. As we shall see below, we can model such transfers and this will change both the configuration of payoffs and which networks emerge.

To get a broader feeling for the ideas of efficiency and stability, let us examine another simple example.

3.2.3 Networks Between Firms: an Industrial Organization Perspective

There are various ways in which firms form relationships that affect market outcomes. They can collaborate in research and development, they can merge, they can produce joint products and ventures, they can contract on specific supplier relationships, they can collude, etc. As the costs of production to various firms and resulting prices and quantities produced and demanded can all vary based on the different relationships between firms, this is a natural setting to apply network formation.[59]

Let us consider an example due to Goyal and Joshi (2003), which allows for easy characterizations of efficient and pairwise stable networks. When two firms form a link it lowers their respective costs of production. This is the only direct effect of a link. There are also indirect effects, as firms eventually compete in the market. The cost structure, and thus the full network structure, affects how much each firm eventually sells on the market and the resulting profits.

In this model the marginal cost of production of firm i is given by $c_i(g) = a - bn_i(g)$, where $n_i(g) = |N_i(g)|$ is the number of neighbors that firm i has in the network g. (Set $a > (n-1)b > 0$ so that costs are always positive.) Thus, each additional alliance that a firm undertakes lowers its marginal cost of production by an amount b.

The eventual profits to firms can be considered under various assumptions about how they compete, with the two canonical ones being pure Cournot and pure Bertrand competition.[60]

[59] see Bloch (2004) for a recent survey.

[60] For non-economists, Cournot competition refers to a situation where producers choose an amount to

Let us first consider Cournot competition, where the market demand is given by the inverse demand function with the price $p = \alpha - \sum_i q_i$, where $\alpha > 0$ is a given constant and q_i is the quantity of the good produced and offered for sale by firm i.

Under the assumption that α is large enough, it is easy to check that each firm's Cournot equilibrium profits are $(q_i(g))^2$, where[61]

$$q_i(g) = \frac{\alpha - a + nbn_i(g) - b\sum_{j\neq i} n_j(g)}{n+1}.$$

From this, as Goyal and Joshi point out, it is very easy to derive the pairwise stable networks. Note that the profits of a firm are increasing in $q_i(g)$. Note also that the network enters $q_i(g)$ in proportion to $nn_i(g) - \sum_{j\neq i} n_j(g)$. Thus, if links have a negligible cost, firm i gains with each link that it adds. If link costs are small enough, then the complete network is the unique pairwise stable network under Cournot competition.

In measuring efficiency here, one might also want to include consumer welfare as well as the payoffs to the firms. The consumer welfare (consumer surplus) is strictly increasing in the total quantity produced, and so they would like to see the complete network formed. As it turns out, the firms' total profits are also increasing in the total number of links formed. Thus, the complete network is efficient whether or not the consumers are accounted for. While the full calculations take a few steps,[62] it is easy to compare the empty network to the complete network. If there are no links formed, then each firm's profits are $\left(\frac{\alpha-a}{n+1}\right)^2$, while if all links form, then each firm's profits are $\left(\frac{\alpha-a+bn(n-1)-b(n-1)^2}{n+1}\right)^2$. Clearly, the total profits are higher when all links form.

Next, let us consider the other textbook form of oligopoly: pure Bertrand competition where the firms charging the lowest price split the market. In this setting, if there are at least two firms who have the lowest cost level, then they will end up bidding their prices down to that cost and splitting the market, but making no profits as the price will equal their cost of production. In contrast, if there is one firm who has a lower cost than the other firms, then that firm will end up capturing the entire market at a price of the second lowest cost level.[63]

produce or a capacity and then the price that clears the market is determined by demand, while (pure) Bertrand competition refers to a situation where firms choose prices and then the lowest priced firm(s) produce to service the entire demand at that price. I will not try to describe these approaches to modeling oligopoly here, as they can be found in most any "principles" textbook.

[61] A firm's profits are $(p - c_i(g))q_i$. The first order conditions lead to $\frac{\partial p}{\partial q_i} q_i + p - c_i(g) = 0$. Noting that $\frac{\partial p}{\partial q_i} = -1$, this implies that in equilibrium $q_i = p - c_i(g)$, and so profits are thus $(q_i)^2$. Solving $q_i = p - c_i(g) = \alpha - \sum_j q_j - c_i(g)$ simultaneously across i, gives the explicit expressions for the quantities. A sufficient condition for all quantities to be positive is that α is large, or that $\alpha - a - (n-1)(n-2)b > 0$.

[62] Consult Goyal and Joshi (2003, 2004).

[63] Working out equilibria in asymmetric Bertrand games has some subtle points if a continuum of prices is allowed. This is because the lowest cost firm would like to underbid the other firms by as small an amount

This makes it quite easy to deduce pairwise stable networks. If there is any positive cost to forming a link, then the only firms willing to form links must be earning a profit. However, the only time any firm earns a profit under pure Bertrand competition is when a single firm has a lower cost than all others, and then only that firm earns a positive profit. This means that at most one firm would ever be willing to bear the cost of a link. Thus, no links will form, and the unique pairwise stable network is the empty network.

Here, we see that the resulting network will not be efficient either from the firms' or the consumers' standpoints. From the industry profit standpoint, it would be better if some links were formed so that some firm earned positive profits (supposing small enough link costs), and in most cases the highest industry profits would actually involve a star network where the center firm would enjoy a very low cost and also see much higher costs and thus high prices from its competitors. This would be the efficient network structure if link costs are small and only firms' profits are considered. From the consumers' perspective, it would be best to see a low price. When the consumers' welfare is also accounted for (and again, link costs are negligible), the efficient network would be one of what Goyal and Joshi (2003) call "interlocking stars". That is where there are two firms, i and j, that are each linked to every other firm, and firms other than i and j are only linked to i and j. This leads to the lowest price and no profits for the firms, but leads to a maximum of consumer surplus (as well as consumer surplus plus profits).

While the networks in these examples again turn out to be stark in their structure, we again see that there are some circumstances where incentives to form links are congruent with overall welfare, and other cases where they are not. In the connections model this depended on the link formation costs. In the above oligopoly models, it is the market structure that determines whether or not there is a tension between stability and efficiency.

3.2.4 A General Tension between Stability and Efficiency

In situations where individual payoffs and welfare are determined by the entire structure of a network, there are naturally externalities present. The decision of some agents to form or sever links can have important consequences for other individuals, who are not directly involved in those links but may be indirectly affected by them. In the connections model, a decision of the center agent in a star to maintain a link with some agent gives indirect benefits to all of the other agents. In the oligopoly model, the decision of one firm to link to another lowers both of their costs, which can be detrimental to the other firms.

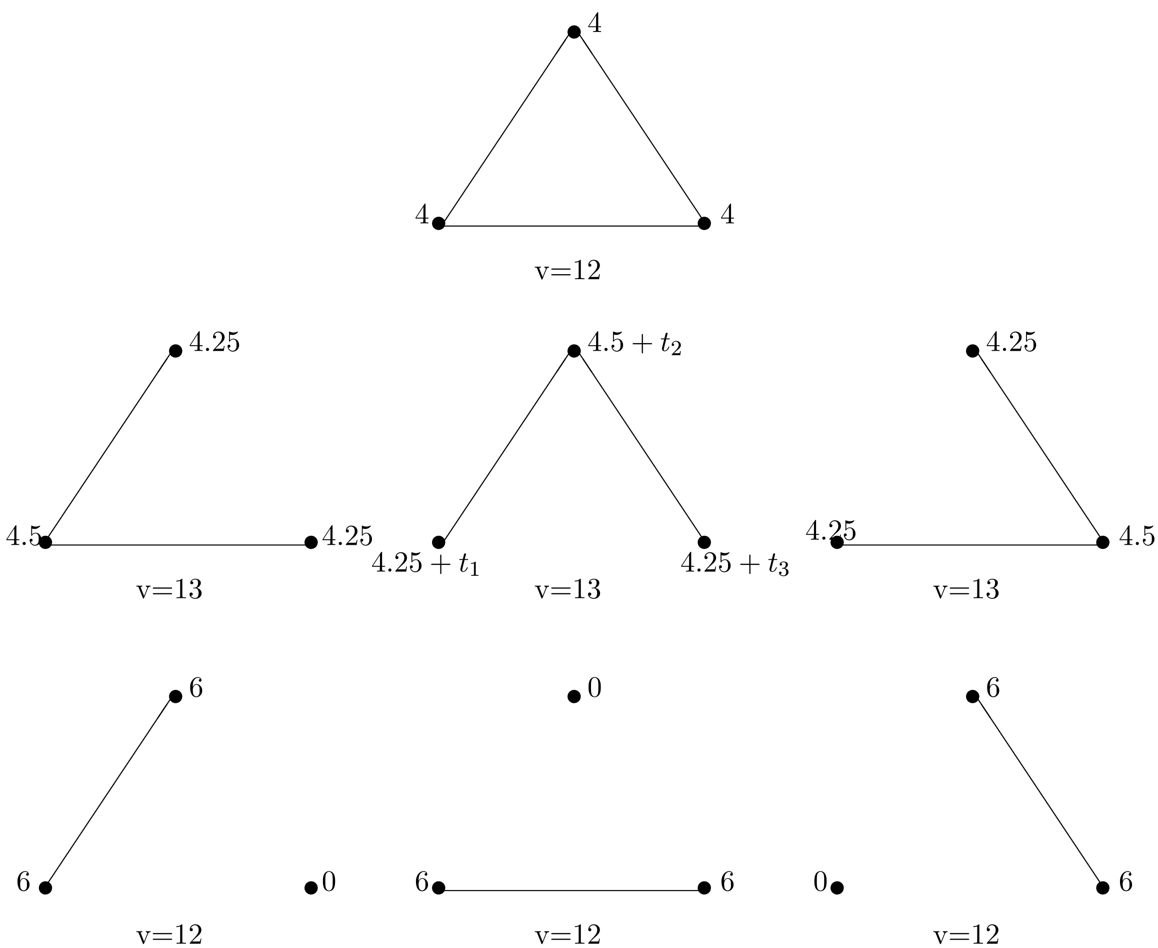
Given that there are externalities present, it should be expected that the networks that

as possible, which means that there are no pure strategy equilibria. However there are equilibria where the higher cost firms mix (with support in a small interval with its min at the second lowest cost), that will lead to the claimed outcome, as described by Blume (2003).

are stable as equilibrium structures do not correspond to those which are efficient. However, what is less expected, is that we cannot always correct this inefficiency, even if we are allowed to tax and subsidize agents for the links they form and even in a complete information setting. The fact that no "reasonable" set of transfers can help rectify the disparity between the equilibrium and the efficient networks is easily seen through the following simple example from Jackson and Wolinsky (1996).

Consider the utilities pictured in the following figure.

The utility of each agent in the complete network is 4. The utility of each connected agent in a linked pair is 6 (with the disconnected agent having utility 0). The efficient network is one with two links, where a total utility of 13 is generated, with the central agent getting a utility of 4.5 and the other two agents getting a utility of 4.25 each.



It is obvious in this example that in the absence of any transfers, the pairwise stable networks all fail to be efficient. The pairwise stable networks are only those involving a single link. In any other network some agent(s) have an incentive to sever a link (every agent has such an incentive in the complete network, and the center agent has an incentive to do so in each of the two link networks).

So, let us consider some possible transfers to try to support an efficient network as being pairwise stable. Given the symmetry of the example, it is enough to consider any of the two-link networks. Let us consider the middle one. We see that the payoffs with potential transfers are $4.25 + t_1$, $4.5 + t_2$ and $4.25 + t_3$. In order to have the transfers be feasible, it must be that $t_1 + t_2 + t_3 \leq 0$. Given the complete symmetry between the first and third agents, let us set their transfers to be equal so that $t_1 = t_3$.

Given that we want to adjust the transfers so as to ensure that the middle two-link network is pairwise stable, we need to make sure that the first and third agents would not gain from adding the missing link. Thus, we need $t_1 = t_3 \geq -.25$. However, in order to have the network be pairwise stable we also need the second agent, or center agent, to be willing to keep both of the links that are in place. As that agent gets a payoff of 6 if either link is deleted, it must be that $t_2 \geq 1.5$. However, now we have violated the feasibility condition as the total sum of transfers needs to be greater than one to ensure pairwise stability.[64]

There are ways around this, but they require treating agents unequally (e.g., setting t_1 and t_3 differently even though the agents are identical in the problem), or else making transfers at some of the other networks in ways that violate some other conditions.[65] For instance, suppose that we set transfers so that we completely equalize utilities for all agents in each network. While this would require allocating utility to agents who may not even be connected to the network, it does provide all agents incentives that coincide with the overall societal value.

3.2.5 Bargaining and Link Formation

Another important point is made by Currarini and Morelli (2000) who show that incorporating the allocation of utilities as part of the bargaining process that accompanies link formation can also lead to more efficient network formation.[66] They describe a specific extensive form game where players announce both the links they wish to form and the payoffs they demand. In that game players are ordered exogenously. Without loss of generality, assume that this is in the order of their labels, so that player 1 moves first, then player 2 and so forth. At his or her turn a player i announces the set of players with whom he or she is willing to be linked

[64] This example extends for weaker notions of efficiency and a variety of notions of stability. See Jackson and Wolinsky (1996) to see extensions to other stability notions, and Jackson (2003) for details on weakening the efficiency criterion.

[65] See Dutta and Mutuswami (1997) for an analysis of ways of reallocating utility so that that some efficient network is strongly stable for a wide variety of settings, when this equal treatment property is dropped.

[66] Mutuswami and Winter (2000) also discuss a similar network formation game and also show that such positive results hold in a broad range of settings, but under a slightly different formulation.

($S_i \subset N \setminus \{i\}$), and a payoff demand $v_i \in I\!R$. The outcome of the game is then as follows. The actions $S = (S_1, \ldots, S_n)$ determine a potential network $g(S)$ by having a link ij be in $g(S)$ if and only if $j \in S_i$ and $i \in S_j$. So a link is in the potential network if and only if both agents involved in the link have said that they are willing to be linked to each other. This is not the final network, as one has to check to see whether the payoff demands that the agents each made can all be satisfied. The network that is eventually formed is determined by checking which components of $g(S)$ are actually feasible in terms of the demands submitted. That is, if g' is a component of g, then g' is actually formed if $\sum_{i \in N(g')} v_i \leq \sum_i u_i(g)$, and otherwise none of the links in g' are formed.[67]

To see how this works in the example in Figure 8, let us suppose that agent 1 moves first, agent 2 moves second and agent 3 moves third. The equilibrium is as follows: 1 announces that he or she wishes to form links with both other agents ($S_1 = \{2, 3\}$) and demands a payoff of $v_1 = 6.5$. 2 then announces that he or she is willing to form a link with agent 1 ($S_2 = \{1\}$) and also demands a payoff of $v_2 = 6.5$. Agent 3 then announces that he or she is willing to form a link with agent 1 ($S_3 = \{1\}$) and demands a payoff of 0. The efficient network $\{12, 13\}$ forms and the payoffs are 6.5, 6.5, 0, respectively. The asymmetry in payoffs is due to the ordering of the players in the game and echoes the point we made before, that completely similar agents must be treated asymmetrically in order to overcome the tension between stability and efficiency.

For the reader interested in the game theoretic details (others can skip this paragraph), we can verify that this is an equilibrium as follows. It is clear that there is no possibility for 3 to earn any higher payoff as the link with 1 is the only link he or she can form (given the other players' announcements) and the link can only be formed if 3 demands a payoff of 0. Let us check on the possible deviations for 2. If 2's deviation leads to a network that ends up involving 1 and 2, then 2 could not demand more than 6.5 and have it be feasible. So, the only possible deviation for 2 that might be profitable, must involve forming the network $\{23\}$. Here, we now need to careful to specify the equilibrium continuation strategies of player 3 in response to player 2's potential deviations. If 2 tries to form a link with only 3 and to demand more than 6.5, then 3's continuation is to form the network $\{13\}$ (by saying that she is willing to form a link with 1 and demanding 5.5) and get a higher payoff since 1 is only demanding 6.5. Thus, 2 has no improving deviation. The remaining deviations to check are those by 1. If 1 asks for $x > 6.5$, then in the continuation 2 can respond with $S_2 = \{3\}$ and anything slightly below x and the network that would form would be $\{23\}$ and 2 would get a payoff above 6.5. Thus, all continuation equilibria after such a deviation by 1 must involve 1 getting a payoff of 0.[68] This equilibrium argument actually gives much of the insight

[67] This game is defined for cases where the utilities are component additive, so that if g' is a component of g and $i \notin N(g')$, then $u_i(g \setminus g') = u_i(g)$.

[68] This also ties down the off-equilibrium path strategies, which are for 2 also to demand x (provided it is

behind Currarini and Morelli's more general argument that efficient networks are the unique equilibrium outcomes in a reasonably broad class of network settings.[69]

These results clearly hinge on the structure of the link-formation-bargaining game, and in particular on its fixed ending point which provides for the asymmetry in bargaining power across the agents (where, for instance, the last player to move is at a real disadvantage).[70] Nevertheless, we have learned that whether or not agents have the ability to bargain over what their payoffs should be at the time of link formation can be important in determining the type of network that forms.

3.2.6 The Economics of Small Worlds

The results from the previous section illustrate some central lessons that have come out of the game theoretic literature, namely that:

- equilibrium networks can differ from efficient networks,
- whether or not efficient and stable networks coincide is context-dependent,
- discrepancies between stability and efficiency can only sometimes be rectified with transfers, and
- the networks that emerge in equilibrium, and the resulting allocation of costs and benefits, depend on various features of the formation process and equilibrium notion.

We also see that the game theoretic analysis has a very different flavor and form than some of the random graph models we discussed in the earlier section. In particular, the predicted equilibrium networks are often quite stark in the nature (stars, complete networks, interlinked stars, etc.). This is partly due to the fact that most of the models that have been solved have strong symmetries in the assumed payoff functions. Without any natural heterogeneity in the problem, it is not surprising that very simple network structures emerge as predictions.

This does not mean that equilibrium models are only suited for deducing broad conclusions about tensions between incentives and efficiency, or other such questions. These models still

no more than 12, and 12 otherwise) and 3 to form the link with 2.

[69] The precise settings are those where the social value of a network is anonymous (so that any permutation of agents in their positions in a network generates the same value), component additive (so the total value of the network is simply the sum of the utilities that each agent would earn if only their component were present), and satisfy size monotonicity (which means that whenever $g + ij$ has fewer components than g, then $g + ij$ results in a strictly higher total utility than g).

[70] See Bloch and Jackson (2003) for an analysis of endogenous transfers in settings that treat players more symmetrically (simultaneous move games). They relate the types of transfers that are needed to reach efficient networks to the types of network externalities that are present in the setting.

have the possibility to provide lessons that are more descriptive in nature. For example, economic forces actually tell us a great deal about *why* we should expect to see "small-worlds". That is, why should we see high clustering on a local level, and short average path length overall? Ideas related to this have been explored in a series of papers (Johnson and Gilles (2000), Carayol and Roux (2003), Galeotti, Goyal, and Kamphorst (2004), Hojman and Szeidl (2004b), and Jackson and Rogers (2005)).

The basic ideas are as follows. Consider a situation where the cost of maintaining a relationship between two agents depends on their proximity. Proximity need not be geographic, but can refer to any sort of nearness according to some traits. It is relatively easier to form friendships when two people attend the same school, have the same profession, or have other things in common. Such low costs on a "local" level help explain why high clustering will be present in a network. The explanation for low average path length in a social network is (slightly) more subtle. Consider a network where costs are related to proximity. Suppose we ended up with a network that exhibited small clusters of individuals who were tightly connected in small groups (those close to each other), and yet the average path length in the overall network was high, due to an absence of links across groups. We might imagine that forming a link that was not "local" in nature was fairly costly. However, with an absence of links across groups, by forming a link that was not "local" in nature one would gain substantial access to a number of agents. The fact that a single link can substantially shorten the distance to a large number of agents at once, is precisely what makes that link valuable. While one will not see as many links that are very costly, the large potential benefit that they bring will mean that they will be present, and that overall distances in the network will have some upper bound.[71]

These ideas are illustrated in the following figure.[72]

[71]This relates to Burt's (e.g., Burt (1992)) idea of structural holes. One will not see too many "structural holes", where the operational definition of a structural hole in this context would be that the addition of a link could substantially shorten distances among two otherwise disconnected or distantly connected groups of nodes.

[72]This figure is somewhat reminiscent of figures concerning Watts' (1999) "caveman" model. However, there is little relationship as the models described here are deriving the structure from equilibrium through an explanation of relative costs and benefits, whereas the caveman model presumes initial clusters and then rewires them.

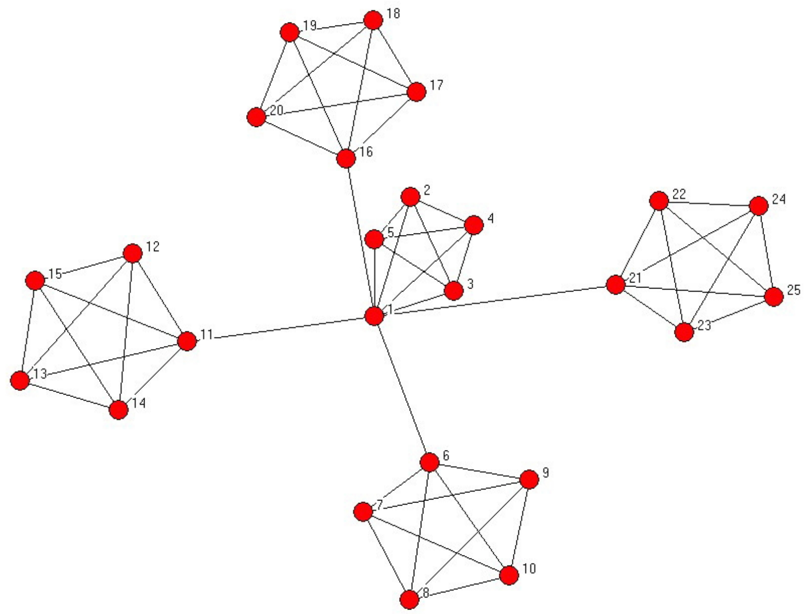


Figure 8: A Pairwise Stable Network in the Islands Model

This is a variation on the connections model called the "islands model" by Jackson and Rogers (2005). In that model, agents are located on separate islands (which might be geographic, professional, or relate to some other characteristics). There is a relatively low cost to linking to an agent on one's own island $c > 0$, while the cost of linking to an agent on a different island is much higher, $C >> c$. The benefits accrue just as in the connections model. Even though the cost is much higher to linking across islands, we still see some links across islands, as if such links were not present, then the gain from linking would be quite substantial as adding one link would provide access to a large number of agents at path lengths of only one or two. In the figure above, it is easy to verify that the given network is pairwise stable when $c < .04$, $1 < C < 4.5$, $\delta = .95$, and where the five agents who are completely connected to each other lie on the same island. While this example is suggestive, it is easy to see that these properties hold more broadly (e.g., see Jackson and Rogers (2005)).

This economic analysis of small worlds gives complementary insights to those of Watts and Strogatz (1998) discussed above, which gives more of an explanation of how it is possible to have both high clustering and short path length at the same time, whereas the above model gives more insight into why we should expect this to be what we see in most social networks. Also, a distinguishing feature between an economic modeling and a random modeling of these features concerns "shortcut" links (i.e., those which link distant parts of the network and if deleted would substantially alter the distance between the connected nodes). In a random model, while unlikely, shortcut links may occur in close proximity to each other. Under the economic approach, the cost of building a second shortcut link next to an existing one would outweigh the benefit.[73]

3.3 Discussion of Models of Network Formation

Let me now discuss some of the strengths and weaknesses of the two approaches to modeling network formation.

An unavoidable challenge in modeling networks is dealing with the complex combinatorial nature of the setting. The number of networks that can form is exponentially large in the number of nodes, which makes tractability a major issue. This has produced substantial hurdles for both the random and strategic approaches to network formation, and yet they have each made remarkable progress in advancing our understanding of what types of networks are likely to emerge.

The random graph based models have a strength of producing specific networks, or distributions over possible networks, which exhibit significant heterogeneity that comes largely from chance and/or through birth dates. The models have provided some insight into how specific features of networks (e.g., fatter tails in the degree distribution) might be traced to certain aspects of a formation process (e.g., some form of preferential attachment). While these models are able to match increasingly long lists of features of observed networks, the processes end up being ad hoc: structured to match those features, and generally we need new processes each time we add a new feature. Another limitation is that the models are descriptions of processes, essentially algorithms, for generating networks. This has two implications. First, as discussed above, this helps answer the 'how" of formation, but does not provide much insight as to the "why".[74] And second, a process does not provide us with methods to evaluate whether the emerging networks are good or bad, that is, whether the resulting networks are efficient.

These last weaknesses are the primary strengths of the game theoretic models. The game theoretic structure provides both a framework for evaluating networks and for understanding why (rather than how) certain networks are likely to emerge. This has resulted in some understanding of the relationship and tension between stable networks and efficient networks. The weakness of the game theoretic approach is that most of the explicit characterizations of equilibrium networks are often so stark that the predicted networks have overly simple structures. Thus, while such models can say something about whether the networks will end up being efficient or not, it has had a hard time predicting things like what sort of

[73] I thank Yann Bramoullé for pointing this out to me.

[74] Another way to phrase this how versus why distinction is to distinguish between reduced form models versus structural models.

degree distribution the network might have. One can push the models to derive some general features, such as those of the "small-worlds" properties discussed in the last section, but deriving very clear pictures of emerging large social networks is still beyond the state of the art.

Interestingly, these two approaches are very complementary. This suggests that some combination of the two approaches might be very fruitful. Incorporating some random elements in terms of which links might be considered at a given time, and then some explicit payoffs and insight into why that link might or might not be added, should end up producing important new insights into the types of networks that we should expect to emerge in different settings, and the how and why behind them.[75]

4 Behavior on Networks

Let me now turn to the second main issue, and discuss a few examples of models of networks that relate social network structure to individual behavior.

As with any scientific study one can be interested directly in the phenomenon itself, or one can be interested in its broader implications. That is, one might simply be interested in (social) networks, and understanding their characteristics, without any broader perspective. As networks are rich and complex by their very nature, they hold much intellectual interest. Nevertheless, we should also be interested in understanding how the networked patterns of social interactions are important in determining (human) behavior and social outcomes. It is this aspect that allows the science and modeling of networks to have relevance outside of itself.

This aspect of modeling networks tends by its nature to be substantially more context specific than the modeling that I have discussed up to this point. That is, relating networks to outcomes relies on examining a setting and understanding the role of the network in that context. This is necessarily context specific, although there are some general tools to be developed and lessons to be learned. As such, it is hard to summarize general findings. Instead here, I will give a few examples that illustrate the variety of applications that have been studied and give a glimpse of things that have been learned.

Let me emphasize here, that understanding how network structure impacts behavior and outcomes is also very important as a building block for the "economic approach" in the

[75] There are some random dynamic models of network formation that are based on incentives to form links, such as Watts (2001), Jackson and Watts (2002a), Tercieux and Vannetelbosch (2004). However, those models use the random process to select from the set of pairwise stable networks, and are thus really more squarely in the game theoretic literature. While these might serve as a starting point, a truer hybrid would involve randomness that really limited the set of potential opportunities to form links in a much stronger way so that some potentially valuable links never even have the chance of being formed.

following manner. The discussion above takes as given that each player can be assigned a payoff as a function of a network. In many contexts those payoffs will be the result of some interaction. For instance, if agents are connected in a network and the network represents trading opportunities, then we must predict as a function of the endowments, preferences, and network structure, which trades will be made, at which prices and how that will determine agents' payoffs. Thus, from primitives and a network structure we can derive induced payoffs and understand how changes in network structure will change the overall efficiency or societal welfare, as well as individual incentives to form or maintain links. We will see more of this below, and it is an essential part of modeling and understanding the impact of social networks.

There is one other aspect of understanding how network structure impacts behavior that also bears discussion. As different structures have different impacts on behavior, understanding them and having an associated cost/benefit or welfare analysis can lead to specific policy prescriptions. For instance, understanding how the centrality of criminals affects their neighbors' criminal behavior has important implications for government policy (e.g., see Ballester, Calvo-Armengol, and Zenou (2003)). Understanding how social networks impact employment opportunities, social mobility, and human capital investments has implications for the subsidization of education (e.g., see Calvo-Armengol and Jackson (2003,2004,2005)).

4.1 Markets and Networks

There is a rich set of studies of markets and networks from an economics perspective (Kirman (1997), Ioannides (1997), Tesfatsion (1997), Weisbuch and Kirman and Herreiner (2000), Kranton and Minehart (2001), Corominas-Bosch (2005), Wang and Watts (2002), Galeotti (2005)), as well as from the sociology literature (e.g., the exchange networks literature following Cook and Emerson (1978)). This is one of the most important and obvious applications of networks to economics as so many markets are not centralized, but rather consist of a complex structure of bilateral trades and relationships.

A recent paper by Kakade, Kearns, Ortiz, Pemantle, and Suri (2004) provides an example of a model that relates market outcomes to random graph-based network structures.[76] They examine a general equilibrium in a market, where the set of trades that can occur are governed by a social network. Their aim is to tie price dispersion to the statistical properties of the underlying network. This is done in the context of a simple buyer-seller model.[77] Buyers have cash endowments and a constant marginal value for a consumption good. Sellers have

[76] Another important example is found in Kirman (1983) and Kirman, Oddou and Weber (1986), who, in the context of core convergence in exchange economies, analyze the impact of limiting blocking coalitions to connected groups, where connection is defined relative to a Bernoulli random graph. Depending on the probability of links forming in the random graph, one can obtain very different conclusions about which coalitions can form and block, and about the resulting core allocations.

[77] See also Kakade, Kearns and Ortiz (2004) for a more general setting.

unit endowments of the consumption good (which they do not value) and desire cash. Buyers thus buy from the least expensive seller(s) with whom they are connected until they have exhausted their cash budget. Prices are seller-specific and determined to clear markets.[78] The full configuration of prices can be quite complex, but the basic intuition is that agents who have more connections should expect better prices and so the price that an agent pays or receives should be related to his or her degree and position in the network. The authors then examine a stochastic process for generating networks of links between buyers and sellers, which is similar to some discussed above (see Section 3.1.5) in that it is a combination of forming links completely at random and forming them in a manner based on preferential attachment.

While the model is difficult to solve analytically, the authors do obtain some bounds in the extremes. For instance, they show that in the extreme where links are formed completely at random, and the probability of forming a link is high enough then there is no price dispersion. In contrast, in the other extreme of pure preferential attachment there will generally be greater asymmetries in the degrees of nodes and there will be price dispersion. Through simulations, the authors then estimate the price dispersion that would result from observed trading patterns based on a United Nations data set of trade volumes.

Given the importance of understanding trade and market structure, this is still an area that deserves much more study. The model discussed above is specific both in its assumptions about transactions and the types of networks it considers, and yet this still proves to be difficult to handle analytically. Moreover, it seems clear that the network structure underlying such trading relationships has a substantial strategic component to it and so the random graph models might not be such good approximations of trading networks, although there is no empirical research to really work from on this question. Previous models based on strategic formation (e.g., Kranton and Minehart (2001) and Corominas-Bosch (2005)) are more tractable analytically, but only represent first-steps in modeling, as they fall short of including the heterogeneity needed (e.g., in endowments, preferences, and production technologies) to match most markets.

4.2 Labor Markets

As discussed in Section 2, it is well-documented that networks of social contacts play an important role in employment. Recent work now brings network structures to the study of employment and wages over time. Calvó-Armengol and Jackson (2004) examine a model

[78] This assumption embodies price-taking, which might be the weakest aspect of the model, given that much of the trading is done bilaterally. Corominas-Bosch (2004) presents an alternative buyer-seller formulation where prices are determined through an explicit bargaining game, and Kranton and Minehart (2001) provide a model where prices are determined through simultaneous auctions.

where agents only obtain information about jobs through a network of connections.[79] Jobs arrive exogenously to the network and agents occasionally lose a job according to some exogenous process. If an agent is already employed and hears about a job, then he or she passes the information on to his or her unemployed neighbors.[80] This passing of information to social ties means that agents' employment and wages over time will depend on their position in the social network, how many social ties they have, and how well-employed those social ties are. They show that this results in correlation patterns in wages and employment of connected agents, and that these patterns depend on the network structure. Also, through simulations, they show that the correlation varies with the distance and location of agents in the network, as well as the structure of the network. They also show that the condition of an agent's social ties has an impact on their decision of whether to stay in the work force or drop out. This results in a contagion effect where if the neighbors of an agent drop out of the labor force, then that increases the likelihood that the agent will drop out, and so forth. This can lead to pockets of drop-outs and persistent unemployment, and among other things, can also help explain persistent differences in wages and drop-out rates across races.[81]

While these results show that incorporating social networks into models of labor markets is important for our understanding of employment and wage patterns, there is still much to be learned about how network structure matters. The empirical and theoretical work to date makes it clear that networks play a key role in labor markets. However, it would be very useful to have a richer understanding of how differences in the structures of agents' social networks impact their wage and employment over time, as well as how the network co-evolves with their career and job choices.

4.3 Learning and networks

Another application of obvious importance in understanding how network structure impacts behavior, is to understand how information propagates through a network, and in particular how different people in a social network learn from each other. Generally, this can be a quite difficult modeling question, as one needs to model information, what it is used for, and how it is observed, transmitted, and/or learned.

Taking a Bayesian perspective is a standard approach in economic modeling, and an obvious starting point. The models of Bala-Goyal (1998) builds from this perspective (see also Allen (1982) and Ellison and Fudenberg (1993, 1995)).

[79] See also Calvó-Armengol (2004) and Jackson and Lopez-Pintado (2005).

[80] See Calvó-Armengol and Jackson (2005) for a richer model where jobs are heterogeneous, the arrival rate may be state dependent, and job information may circulate indirectly through the network.

[81] Calvo-Armengol and Jackson (2005) study how investments in education based on social network status can also help us to understand the prevalence of social *immobility*, which has been found without exception in countries around the world.

Bala and Goyal (1998) make a very simple but important point. Consider a series of agents connected in a social network who all face the same stationary, but random, environment. The network is fixed and time progresses in discrete dates where agents each choose one of a finite set of actions at each date. The payoffs to the actions are random and their distribution depends on an unknown state of nature. The agents are all faced with the same set of possible actions and the same unknown state of nature. They all have identical tastes and face the same uncertainty about the actions. Over time, each agent observes his or her neighbors' choices and outcomes. The main conclusion is that eventually the agents will converge to choosing the same action, based on the observation that over time players who observe each others' actions and payoffs should eventually come to choose the same action.[82]

For example, consider a situation with two choices A or B. A results in a payoff of 1 per period for certain. B pays 2 with probability p and 0 with probability $1-p$. So, if $p > 1/2$ then any agent would prefer to choose B (given either sufficient patience or risk-neutrality). However, p is unknown to the agents. What Bala and Goyal show is that each agent in a connected network will obtain the same long-run utility. The intuition is as follows. We need only reason that any two neighbors earn the same long run utility, as this implies the same must be true network-wide. If one neighbor is doing better than another, then the neighbor with the poorer action, will observe the other agent. For instance, suppose that one agent is choosing A each period and the other is choosing B and $p > 1/2$. The agent choosing A will learn that $p > 1/2$ by observing the other agent's payoffs and will eventually switch to choose B as well.[83]

Note that the fact that all agents end up with the same long run utility does not mean that all agents converge to choosing the "right" action. It is possible that all agents start out choosing A and always choose A (being sufficiently pessimistic about B to make it not worthwhile to even experiment), even though B would lead to a higher payoff. However, Bala and Goyal show that if the network is large enough, and there are enough agents who are optimistic about each action spread throughout the network, then the probability that the society will converge to the best overall action can be made arbitrarily close to 1. The idea is that there will be sufficiently many experiments by the optimistic agents so that the true payoff of each action will be learned and then the society will converge to the right action.

While the above lessons show the potential for the long-run conductance of information through a network, they do not give us much impression of what happens in the shorter

[82] See Morris (2000) for another analysis of the spread and convergence of behavior through a network, but in a different context where uncertainty regards strategic choices of others and players care about their neighbors' choices.

[83] Bala and Goyal work with a boundedly rational model. See Gale and Kariv (2003) for a Bayesian analysis. Also, see DeMarzo, Vayanos, and Zwiebel (2003) for a setting where beliefs are updated over time in a boundedly rational way, but where they need not converge across agents, as actions are only taken once and it is only information that is repeatedly passed.

run, which might often be quite relevant, especially if the world is not stationary. Nor does network structure enter the above discussion in any meaningful way. There are papers that have made more progress on understanding how network structure impacts beliefs. Gale and Kariv (2003) (see also Choi, Gale and Kariv (2004) and Celen, Kariv, and Schotter (2004)) explore the interaction between network structure and beliefs under a variety of learning assumptions. Due to the complexity of some of the inference problems, they are able to provides a detailed understanding for long run beliefs in small networks (e.g., three nodes) but leave open questions regarding more complex networks. DeMarzo, Vayanos, and Zwiebel (2003) are able to deal with more general network structures by assuming that agents follow a specific belief updating rule, where agents (erroneously) treat new iterations of information as independent of previous iterations. They document an intuitive relationship between the position of an agent in the network and their resulting impact on beliefs and opinions. These studies are important steps in developing a fuller understanding of how interaction structure affects information dissemination and belief formation.

4.4 Spread of Information, Viruses, Disease

Related to studies of learning through a network, which have focused on belief updating and action choice, there are also studies of the physical spread or transmission of infections and behavior that are transmitted directly or by chance, and not through some updating or optimization procedures. Examples include the spread of diseases, computer viruses, and also the spread of some types of behaviors, beliefs, and information. Standard models of such spreading come from the epidemiology literature, which has focussed on the spread of contagious disease. One model that is useful to discuss a bit is the SIS model ("susceptible, infected, susceptible" model, see Bailey (1975)), which is a variation on the seminal model in the literature, the SIR model ("susceptible, infected, removed" (SIR) model, which dates to Kermack and McKendrick (1927)). Such models were originally based on random meetings of individuals. However, networked interactions were discussed as early as Rapoport (1953, 1953b), and eventually models that allow for network structure were studied by Anderson and May (1988) and Sattenspiel and Simon (1988). How infection rates depended on specific aspects of the network structure has more recently been studied by Kretschmar and Morris (1996), Pastor-Satorras and Vespignani (2000, 2001, 2002), Lopez-Pintado (2004), and Jackson and Rogers (2004), among others. In particular, these studies allow one to estimate infection rates based on degree distributions.

Let me describe this setting in a bit more detail, as it offers one of the clearest understandings of how network structure can be related to outcomes, and the tools and methods used in such analyses look to be useful in other contexts.

Consider a network where a given healthy (also called "susceptible") node catches a disease in a given period with a probability $\nu k_i f_i$, where $\nu \in (0,1)$ is a parameter describing a rate of transmission of infection in a given period, k_i is the (in-)degree of node i, and f_i is the fraction of i's neighbors who are infected.[84] Also suppose that any "infected" node recovers in a given period with a probability $\delta \in (0,1)$. Thus, nodes are either susceptible or infected, and can alternate between these states depending on the state of their neighbors. This results in a Markov chain. We can then ask a series of questions. First, how high does the infection rate ν have to be relative to the recovery rate δ in order to have the infection reach some nonzero steady state in the population? Second, can we estimate the long-run steady-state proportion of infected nodes? Third, can we relate the answer to these questions to the network structure?

The heuristic (mean-field-based) approach that has been used in this literature to estimate infection patterns is as follows. Consider a large network whose degree distribution is described by P, where $P(k)$ is the proportion of nodes that have degree k. Moreover, let us make the (restrictive) assumption that there is no correlation in degree between linked nodes. Let $\rho(k)$ denote the steady-state infection rate of a node with degree k, and ρ be the average across nodes: $\rho = \sum_k \rho(k)P(k)$. The probability that a given link points to a node of degree k is $\frac{kP(k)}{<k>}$ (where $<k>$ is the average degree under P, $<k>= E_P[k]$). Note that this is different from the distribution of degrees across nodes, as nodes with higher degree are proportionally more likely to be reached via any given link. Using this we estimate the probability that a given link points to an infected node in any given period in a steady state distribution under the mean-field hypothesis. This is represented by the parameter

$$\theta = \frac{\sum \rho(k)kP(k)}{<k>}. \tag{5}$$

Now to estimate the steady-state value of $\rho(k)$, we set the change in the proportion of nodes of degree k that are infected to 0. That is,

$$0 = \frac{d\rho(k)}{dt} = \nu k\theta(1-\rho(k)) - \delta\rho(k).$$

Here, $\nu k\theta(1-\rho(k))$ represents the number of healthy nodes that become infected, and $\delta\rho(k)$ represents the number of sick nodes that become healthy. These must be equal in a steady state.[85]

[84] This is obviously a fairly specific infection mechanism, but for small infection rates offers a reasonable approximation of the probability of getting infected if infection rates are independent across neighbors (a questionable assumption if there is clustering). See Lopez-Pintado (2004) for the analysis of other infection mechanisms.

[85] The system here is clearly heuristic, as if we ran a Markov process on this system, all nodes would eventually converge to being susceptible and this is an absorbing state. To be careful, one needs to have some exogenous probability that the nodes become infected even when none of their neighbors are.

Letting $\lambda = \nu/\delta$, we derive

$$\rho(k) = \frac{k\lambda\theta}{k\lambda\theta + 1}. \tag{6}$$

Equations 5 and 6 can the be solved simultaneously to derive the steady state distributions.

Let us first solve this in the easy case where the network is completely regular so that all nodes have degree $< k >$. In that case, $\theta = \rho$ and (6) becomes

$$\rho = \frac{< k > \lambda\rho}{< k > \lambda\rho + 1}.$$

There is always a solution of $\rho = 0$, and then also another solution of $\rho = 1 - \frac{1}{\lambda <k>}$ which is greater than 0 only if $\lambda > \frac{1}{<k>}$. Thus, in order for infection to spread in the network, the relative infection/recovery rate has to exceed a threshold. This is another example of a phase transition, of which we saw examples earlier in the discussion of the properties of random graphs.

Pastor-Satorras and Vespignani (2000) solve this system in the case where the degree distribution is scale free (using $P(k) = 2 < k > k^{-3}$). They find an approximation of $\rho = 2e^{-1/<k>\lambda}$ (for small λ). Thus, they deduce that even with tiny values for λ, there will be some non-zero infection rate in a scale-free network. This contrasts with the fact that λ has to exceed a positive threshold in a regular network in order to reach a non-zero infection rate.

Lopez-Pintado (2004) uses the following approach to characterizing situations where the solution for the steady state θ (and thus the steady state infection rates $\rho(k)$ and ρ) will be nonzero.[86] Let

$$H(\theta) = \sum \frac{kP(k)}{< k >} \left(\frac{\lambda k\theta}{\lambda k\theta + 1} \right). \tag{7}$$

So fixed points of H correspond to steady-state distributions. Note that $H(0) = 0$, and that H is increasing and strictly concave in θ. Thus, in order for H to have another fixed point above $\theta = 0$, it must be that $H'(0) > 1$.[87] Let us check when this is true. Note that

$$H'(\theta) = \sum \frac{kP(k)}{< k >} \left(\frac{\lambda k}{(\lambda k\theta + 1)^2} \right).$$

That is, $H'(0) = \lambda E_P[k^2]/ < k >$. Thus, in order to have $\theta > 0$ (and thus a steady-state infection rate $\rho > 0$), we must have $\lambda > \frac{<k>}{<k^2>}$ (where $< k^2 >= E_P[k^2]$). In the regular network, this is the claimed threshold of $1/ < k >$, while in a scale-free network $< k^2 >$

[86] See also Pastor-Satorras and Vespignani, (2001, 2002). Moreno, Pastor-Satorras and Vespignani (2003) also allow for correlation among degrees in a family of distributions.

[87] Noting that H is continuous and increasing in θ, $H(0) = 0$, and $H(1) < 1$ (from equation (7), as this is the expectation of an expression that is always less than 1), it follows that there will be a fixed point above 0 whenever $H'(0) > 1$.

is infinite and so the threshold is 0. For a Poisson degree distribution it falls somewhere between the two extremes.

The basic idea is that nodes with high degree can serve as a conduit for infection. Even very low infection rates can lead them to become infected as they have so many neighbors. They then can pass the infection on to a large number of nodes. The degree distribution then determines the relative makeup of the network in terms of nodes of different degrees. In a regular network, every node has the average degree. As we move to a Poisson distribution, we begin to see more of a spread in the distribution and some higher degree nodes and others of lesser degrees. This leads to a lower threshold at which infection can be sustained, as the higher degree nodes can begin to serve as the conduits as discussed above. As we continue to increase the spread and move to a scale-free network, we have extremely high degree nodes, and very low degree nodes. The variance is in fact infinite and infections can be sustained at arbitrarily low net rates of contagion.

Building on this methodology, Jackson and Rogers (2004) show one can completely order both the threshold rates of contagion needed to sustain an infection and the resulting infection rates in terms of the networks degree distribution, by ordering the distributions in the sense of second order stochastic dominance.[88] In terms of the thresholds for infection, it is clear from the above that as we change a network in the sense of second order stochastic dominance, then we increase $< k^2 >$ and so we decrease the threshold λ needed for an infection to be sustained ($\frac{<k>}{<k^2>}$).

What is a bit more subtle is that the behavior of the steady state distributions as it relates to network structure exhibits very different features at low levels of λ than it does at high levels. Jackson and Rogers (2004) show the following:

Consider two networks with degree distributions P and P', respectively, where P second order stochastically dominates P', and both have the same average degree.[89] Then there exist $\underline{\lambda}$ and $\overline{\lambda}$ such that

- If $\frac{\nu}{\delta} < \underline{\lambda}$ then the steady-state average infection rate under a mean-field approximation is lower under P than P'.
- If $\frac{\nu}{\delta} > \overline{\lambda}$ then the steady-state average infection rate under a mean-field approximation is higher under P than P'.

[88] One can also order things in terms of first order stochastic dominance, but that relationship is quite obvious. If we simply increase the overall number of links then we will increase infection rates and decrease the threshold needed to sustain infection.

[89] If P and P' have the same average, then P second order stochastic dominates P' if and only if P' is a mean preserving spread of P, which is equivalent to having $\sum_k f(k)P(k) \geq \sum f(k)P'(k)$ for all concave functions f. This implies that P' has a (weakly) higher variance than P, but also requires a more structured relationship between the two.

These results are illustrated in the following figure, which pictures infection rates for three varieties of networks (holding average degree constant).

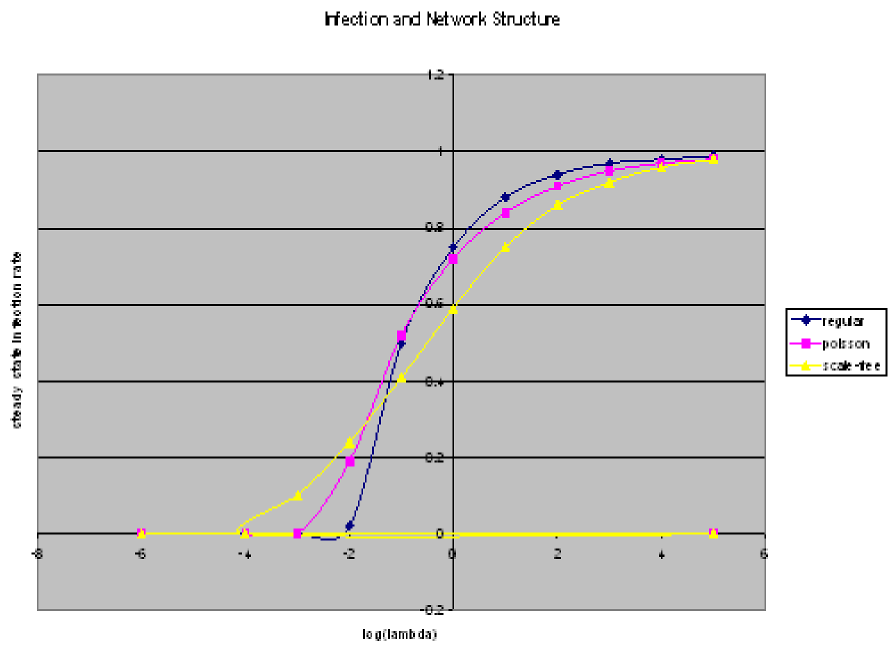


Figure 9: Steady State Infection Rates for Three Degree Distributions, as a Function of the Log of the Relative Contagion/Recovery Rate

The intuition behind these results can be expressed as follows. The change in infection rate due to a change in network structure (and in particular, a change in the degree distribution) comes from countervailing sources, as more extreme distributions have relatively more very high degree nodes and very low degree nodes. As discussed above, very high degree nodes have high infection rates and serve as conduits for infection, thus putting upward pressure on average infection. Very low degree nodes have fewer neighbors to become infected by and thus tend to have lower infection rates than other nodes. As we make a mean-preserving spread in the degree distribution, the overall impact depends on how the (direct and indirect) increase in infection in changing some nodes to have higher degree compared to the decrease in infection due to changing some nodes to have lower degree. When infection rates are already high, infection rates tend to increase less than linearly in the degree of a node (if simply due to the fact that they cannot increase above one). While if infection rates are low, then there is a more than offsetting increase in infection due to an increase of some nodes' degrees, as their increased degree not only increases their infection rates, but also leads to an increase in transmission.

What we learn from this analysis extends far beyond the understanding of infection rates. It shows how we can use statistical characteristics of networks (e.g., comparing them in terms of stochastic dominance of degree distributions) to deduce the impact that they have on behavior or outcome. These same methods have the potential to be applied much more broadly to deduce how network structure impacts behavior in many other settings.

4.5 Public Goods

Another setting where we can see how network structure influences outcomes is the provision of local public goods.

For instance, consider a model analyzed by Bramoullé and Kranton (2005). Agents each choose an effort level $e_i \in [0, \infty)$. The agents are connected in a network, and they derive benefits not only from their own effort level but also from their neighbors - that is, the other agents to whom they are directly linked. For instance, think of agents each collecting information and then sharing that information with their direct neighbors. This includes a variety of applications, from consumers sharing information with their friends about products they have tried, to companies sharing information about production processes, to researchers sharing ideas and methods. For simplicity, the model only considers the benefits flowing to direct connections.

Agent i's utility in a network g when efforts $(e_1, \ldots, e_n)$ are exerted is

$$u_i(g, e) = b(e_i + \sum_{j \in N_i(g)} e_j) - ce_i,$$

where b is a continuously differentiable strictly concave function and $c > 0$ is a cost parameter.

Supposing that the solution $b'(e^*) = c$ is well-defined and has $e^* > 0$, a great deal can be deduced about the structure of the equilibria. Given the payoff structure, it is clear that each neighborhood will have a total of at least e^* produced. Normalizing $e^* = 1$, this could happen in various ways:

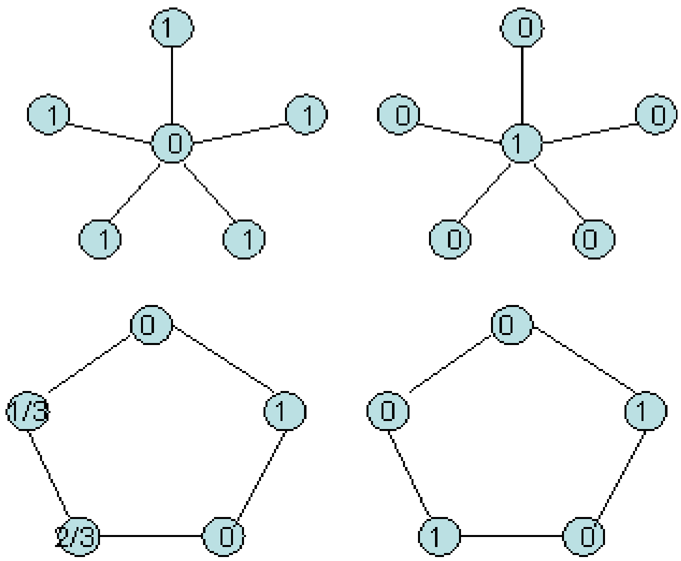


Figure 10: Examples of Equilibrium Public Good Provision Choices in Two Networks

There is a sense in which the equilibria where some agents specialize and provide effort of 1 and others provide an effort of 0 are more robust than the others.[90] Bramoullé and Kranton (2005) refer to these as specialized equilibria.

There is a complex structure to specialized equilibria, and there is a multiplicity of them. For instance, consider any maximal independent set of nodes.[91] Then have each node in the maximal independent set choose action e^* and all nodes outside of the set choose 0. As Bramoullé and Kranton point out, this is clearly an equilibrium, and moreover, all specialized equilibria must be of this form.

As this sort of public good model captures the substitutability of actions of neighbors that would apply to many settings, it provides a particularly interesting one for further study. The multiplicity of equilibria that Bramoullé and Kranton (2005) note, provides a hurdle in terms of making predictions about how network structure affects behavior, but there are well-defined ways in which some equilibria appear to be more natural or robust than others, and it appears that there is much that can be said about how behavior relates to structure in the context of large networks.

[90] See Bramoullé and Kranton (2005) for details. They examine a perturbation where each agents actions can be perturbed by a small amount and then must have a best response process converge back to equilibrium. The only robust equilibria in this sense are specialized equilibria in which each non-specialist agent has at least two specialists in his or her neighborhood.

[91] An independent set of nodes is a set such that no two nodes in the set are connected. A maximal independent set is an independent set which is not a strict subset of any other independent set. It is then easy to see that a maximal independent set is a set such that no two nodes inside the set are connected to each other and any node outside of the set is connected to at least one node inside the set.

4.6 Other Topics

There are many other areas that have been studied that relate network structure to outcomes, and ones that are of obvious social and economic relevance. For instance, Ballester, Calvo-Armengol, and Zenou (2004) examine how network structure influences criminal and other behavior in a model where there are local positive externalities (for instance learning or receiving help from friends that are also criminals) and global negative externalities (for instance overall competition). Within a class of such models they are able to completely characterize equilibrium outcomes and relate these to a measure of path-centrality in a network. This provides interesting new insights relating behavior to centrality in a network. Activity by players who are more central has more impact on other players' level of activity and leads to greater feedback effects.

There are also recent studies of risk-sharing on networks, which build on evidence from recent studies (e.g., Fafchamps and Lund (2003) and De Weerdt (2002)) that indicate that network structure plays a major role in determining how well risk is shared in rural societies. Theoretical studies have looked at two issues: how the network structure can be modified over time as an endogenous part of the risk sharing (e.g., see Bloch, Genicot, and Ray (2005)) as well as how the network structure affects the equilibrium incentives (e.g., see Bramoullé and Kranton (2005b)).

There are also studies of play in games with complementarities in neighbors' actions, such as in the context of coordination games (e.g. Ellison (1993), Young (1998), Morris (2000), Jackson and Watts (2002b), Droste, Gilles, and Johnson (2000), Goyal and Vega- Redondo (2005), Feri (2003), Lopez-Pintado (2005)) as well as other structures with complementarities (e.g., Galeotti and Vega-Redondo (2005)). These studies have looked at both the change in play and co-evolution of the network itself. However, to date, most of the work has focussed on very simple games (e.g., two-by-two coordination games), and there is much that is unknown beyond these special cases.

The wide variety of settings where network structure is an important determinant of behavior makes it clear that this is one of most wide open and important areas for further study.

5 Whither Now?

As we have seen, there is much that we know about the structure and use of social networks, and a growing set of models to describe their emergence, roles, and importance in determining social outcomes. At the same time, as alluded to at many points in the discussion so far, there is so much that we have yet to understand or model. Let me provide a partial list of some of what I see to be the most obvious and pressing issues for study.

- The above discussion of how network structure affects behavior illustrates that despite the difficult combinatorics faced in many applications, there is still much that we can deduce. Moreover, the wide variety of settings where social networks play a role leads to an almost endless set of interesting avenues to investigate. This continues to be one of the more promising areas for the modeling of social networks, and should also prove to be one of the main interfaces between theoretical and empirical work.

- One of the main points I made in the discussion of modeling of network formation is that there is a great potential to combine ideas from random graph models of network formation with those strategic formation models. These are largely complementary models and there look to be substantial gains in producing hybrids, both in terms of providing better fits of observed networks and leading to a better understanding of the tension between stability and efficiency.

  In many applications, it seems that an appropriate model would be one where the potential opportunities to form links or social ties is where the randomness plays its major role. This randomness might have structure to it, as for instance the opportunities that we have for scientific collaboration are partly determined by where we work, whom we have worked with in the past, which conferences we attend, coupled with a good amount of chance meetings and conversations. At the same time, which opportunities appear to be worth pursuing, and which ties and friendships we maintain over time, owe much to the costs and benefits that they provide. Building models that tractably combine these two features could also help produce models that lead to a deeper understanding of the dynamics of networks and social relationships over time, something that is still largely absent from the existing literature.[92]

- This also leads us to another aspect of network formation models that could be improved upon. Existing models generally deal with link formation where the action takes place at the link level. That is, either links are being randomly determined, or agents are deciding which links they would like to form. However, in many social settings, basic decisions are made that determine large sets of links all at once. For instance, the decision of which university to attend determines one's classmates and the decision of where to work determines one's colleagues, en masse. Models that deal with how such larger decisions impact social network structure are virtually nonexistent, and yet this is an essential part of what is often implied by the term "networking," as used in the

[92] Jackson and Watts (2002) provide an evolutionary model where chances to form links are random, but links are then formed strategically. However, they focus on the limit where the randomness becomes negligible. Understanding the case before the limit, might serve as a basis for one model that encompasses randomness and strategic choice.

vernacular regarding building social relationships.[93]

- There are also many facets of existing network formation models within the existing separate strands of random graph and game theoretic models that deserve further study.

  - With regards to models of strategic formation, there are questions of how to appropriately model equilibrium, given that the consent of parties might be needed to form links. Are social relationships considered one at a time, or many at once, and how does this depend on the setting? How coordinated are the decisions among groups of agents? To what extent are bargaining and/or transfers part of the process? Do agents take into account the impact their decisions will have on the further evolution of the network? These questions have led to some recent research on the foundations of modeling strategic network formation[94], of which a deeper understanding is needed.
  - With regards to random networks, two things come to mind. First, it is clear that there is great need for more detailed structural fitting of the models and that this might help in the development of new models. For instance, as discussed above, "scale-free" networks are at best an idealization and benchmark and that only some observed degree distributions exhibit such features, and even then only approximately and in one tail. Second, social networks exhibit much richer heterogeneity in the types of interactions and have correlation structures that are not adequately captured by existing models. In particular, things such as ethnicity, profession, and geography, produce neighborhood structures that are different from those predicted in existing models. For example, Adamic (2005) provides an interesting analysis of the cross-citation by political blogs. There are distinct separations between different political ideologies, with inter-ideological linking arising on special occasions. Existing models have not really captured such features, and it is clear that such features should be very important in influencing opinion.[95]

- Another area that is crying out for attention, and just beginning to receive it, is the modeling of the strength of ties. The idea that social relationships are not 0-1 in their nature is quite clear, and was the center piece of one of the most influential social networks papers - Granovetter's (1973) article on the "strength of weak ties." Granovetter

[93] See Ioannides and Soetevent (2005) for a first step in this direction. I should also mention that there is a rich literature on coalition formation (e.g., see the book by Demange and Wooders (2004) and references therein), which could end up playing a role in these developments.

[94] See, for instance, Calvo-Armengol and Ilkilic (2004), Bloch and Jackson (2004), Tercieux and Vannetelbosch (2004), Ilkilic (2005), Slikker and van den Nouweland (2005), Chakrabarti (2005), Page and Wooders (2005).

[95] See Watts, Dodds and Newman (2002) for a model in this direction.

pointed out the importance of weaker social relationships (according to various measures of the level and/or frequency of interaction) in providing critical information. This work has produced volumes of empirical studies (e.g., see Granovetter (1995) and some of the references therein). Yet, until recently there were only a few models (Boorman (1975) and Montgomery (1991, 1992, 1994)) of how networks might form when both weak and strong ties are possible, and both of these models were specialized to the context of job contact networks. Recent interest in this subject has resulted in more general studies (e.g., Rogers (2005), Bloch and Dutta (2005)) that investigate models where agents decide on how much effort or time to devote to their different relationships. Such models look to provide interesting insights regarding how complementarity or substitutability affects the patterns of efforts that ensue. A related issue is that relationships vary not only in their intensity, but also in their patterns over time. Interaction patterns tend to be sporadic. Understanding some of the timing of interactions, and more basically what is entailed in a relationship, might help shed better light on the differences between things such as strong and weak ties and how they differ across applications.

- As mentioned at the beginning of Section 3.2, the early models of how the allocation of total value or benefit among players depends on a network structure emerged from the cooperative game theory literature. The perspective that cooperative game theory (even with graph-restricted games) gives to this problem is not always rich enough to address the issues that arise in a social network context (e.g., see Jackson (2005) for a discussion of this point). It is also clear, that the formation of social relationships often involves some bargaining. For instance, dowries are an obvious example. We have seen above how such bargaining can be instrumental in determining how efficient the resulting network structure is. Yet, the models that we have, both from a cooperative and a non-cooperative approach, are still far from giving us a full understanding of how value is shared among members of a social group, how this is determined by the network structure, and how this affects network formation.

- As the investigations in network analysis continue to multiply, so does the need for well-understood tools and methodology. For example, there are numerous measures of how "central" a node is in a network, ranging from simple comparisons of node degrees to detailed analyses of the eigenvalue structure of modified adjacency matrices. Such different measures are clearly identifying different facets of a node's role in a network. However, our current understanding of which (if any) existing measure is appropriate in which context comes almost entirely from seeing how measures operate on various examples and then judging which seems to be capturing what we are after.[96][97]

There are numerous other concepts in social network analysis that are associated with a variety of definitions and measures, and little to guide us in terms of understanding of the various properties that they possess. I am only aware of a few studies that examine the properties of different measures and definitions used in social network analysis,[98] while the extent of social network science clearly requires more.

- It is clear that the links that people maintain are different in nature. For example, they might relate to friends, relatives, co-workers, neighbors, or casual acquaintances. Each type of link might be active or useful under different circumstances, and might involve different costs and benefits. Modeling the interaction between different overlapping network structures could potentially lead to new insights into things such as the dissemination of information throughout a population. For example, as Granovetter (1974) notes: "... much of the information about jobs that one receives through contact networks is a byproduct of other activities, and thus not appropriately costed out in a rational calculation of the costs and benefits of getting information."[99]

- As predictions from models continue to proliferate, experiments will provide an increasingly important testbed.[100] This is especially true of things like the dissemination of information, which can be very difficult to pinpoint outside of the controlled environment of a laboratory.

- The rich collection of case-studies from the sociology literature is quite remarkable partly because of the level of difficulty that researchers have historically faced in iden-

[96] See Borgatti (2003) for one study which tries to sort through some centrality measures.

[97] The same is true of identifying community structures in networks. A community structure is a partitioning of the nodes of a network into groups or "communities", with the idea that the nodes in the same community are somehow similar or equivalent according to some criterion. This can be useful in terms of simplifying a complex network into a simpler smaller network of relationships between communities, as well as learning something from the network relationships regarding which nodes are similar. Many algorithms have been developed to identify community structures in networks, and yet most of what we know about the relative merits or deficiencies of various approaches and algorithms comes simply from examining whether they seem to give the "right" community structuring in various examples. An overview of some of this literature can be found in Newman (2004b) and a discussion of the importance of identifying the properties for identifying community structures can be found in Copic, Jackson and Kirman (2005).

[98] For example, see van den Brink and Gilles (2000), Gomez et al (2003), Monsuur and Storcken (2004), and Copic, Jackson and Kirman (2005).

[99] I thank Yann Bramoullé for bringing this quote to my attention.

[100] Here I am referring to experiments with human subjects, although the term experiments is also used in some literatures to refer to simulations, often involving agent-based modeling techniques. These too are an increasingly relevant tool as technology now enables us to work with much richer models in simulations than we can handle analytically. See Kosfeld (2003) for some discussion and references.

tifying network structures. This often involved interviews of subjects, or careful observation of some group over time, and limited both the scope and quality of the data that could be collected. Advances in both telephony and internet communication (including email), as well as computing technology, has recently made readily available large, detailed, and precise interaction patterns; which in some cases are less prone to measurement error and easier to work with. Moreover, such data sets give new dimensions to network structures as they have detailed time-stamps with which to study the dynamics of interaction. This greatly enhances the potential for empirical testing of increasingly complicated network models, and should also enrich the stable of questions for models to address.

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