

## CSE 528

### Lecture 11: Recurrent Networks (Chapter 7)

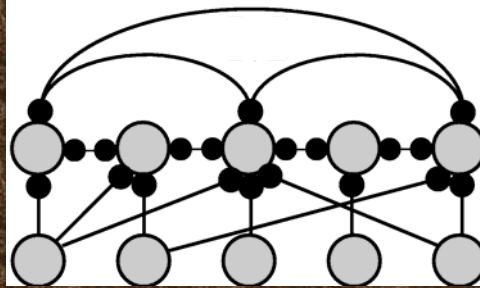


Image from <http://clasdean.la.asu.edu/news/images/ubep2001/neuron3.jpg>  
Lecture figures are from Dayan & Abbott's book  
<http://people.brandeis.edu/~abbott/book/index.html>

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### What's on our platter today?



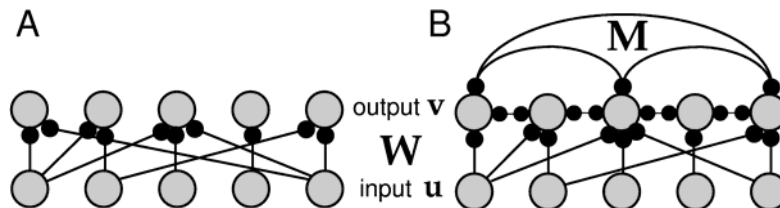
- ◆ Computation in Recurrent Networks
  - ⇒ Linear Recurrent Networks
    - ◆ Stability analysis using eigenvalues
  - ⇒ Nonlinear Recurrent Networks
    - ◆ Can amplify inputs
    - ◆ Can select inputs
    - ◆ Can multiply (gain modulation)
    - ◆ Can store short-term memory
  - ⇒ Associative Memory (Hopfield net)
    - ◆ Showing Stability via Lyapunov function
  - ⇒ Oscillatory networks

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## Feedforward versus Recurrent Networks

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$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathbf{W}\mathbf{u} + \mathbf{M}\mathbf{v})$$

Output    Decay    Input    Feedback

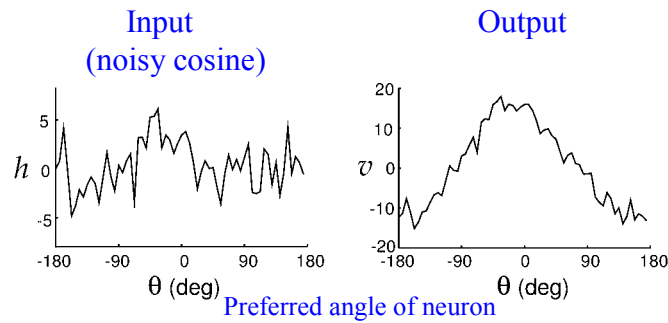
(For feedforward networks, matrix  $\mathbf{M} = 0$ )

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## What can a Linear Recurrent Network do?

Analysis based on eigenvectors of recurrent weight matrix

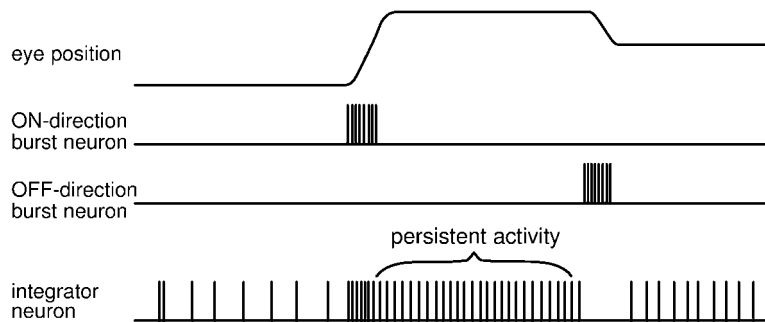
## Amplification in a Linear Recurrent Network



$$M(\theta, \theta') \propto \cos(\theta - \theta')$$

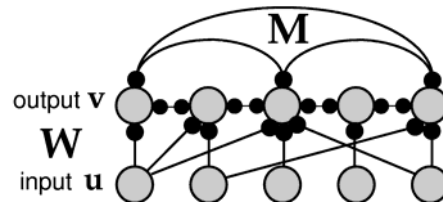
All eigenvalues = 0 except  $\lambda_1 = 0.9$  i.e. amplification =  $\frac{1}{1 - \lambda_1} = 10$

## Input Integration for Maintaining Eye Position



**Input:** Bursts of spikes from brain stem oculomotor neurons  
**Output:** Memory of eye position in medial vestibular nucleus

## Nonlinear Recurrent Networks



Two types of firing-rate models

$$\tau \frac{d\mathbf{I}}{dt} = -\mathbf{I} + \mathbf{W}\mathbf{u} + \mathbf{M} \cdot F(\mathbf{I})$$

Current Dynamics  
(firing rate  $\mathbf{v} = F(\mathbf{I})$ )

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathbf{W}\mathbf{u} + \mathbf{M}\mathbf{v})$$

Firing-Rate Dynamics

Output Decay    Input    Feedback

(Convenient to use  $\mathbf{W}\mathbf{u} = \mathbf{h}$ )

## Continuous Nonlinear Recurrent Networks

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathbf{h} + \mathbf{M}\mathbf{v}) \text{ or,}$$

$$\tau \frac{dv_i}{dt} = -v_i + F\left(h_i + \sum_j M_{ij} v_j\right)$$

Discrete case  
(small number of neurons)

Continuous case (in the limit of large numbers of neurons):

$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + F\left(h(\theta) + \rho_\theta \int_{-\pi}^{\pi} M(\theta, \theta') v(\theta') d\theta'\right)$$

$\theta$  = preferred stimulus of the neuron (e.g. orientation of input)

## Example of a Continuous Recurrent Network

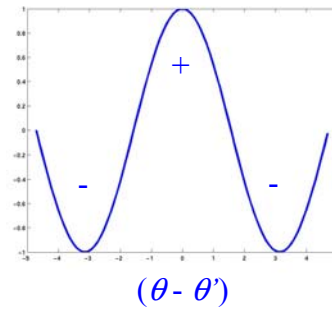
Choose  $F =$  rectification nonlinearity:

$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + \left[ h(\theta) + \frac{\lambda_1}{\pi} \int_{-\pi}^{\pi} M(\theta, \theta') v(\theta') d\theta' \right]^+$$

$$M(\theta, \theta') = \cos(\theta - \theta')$$

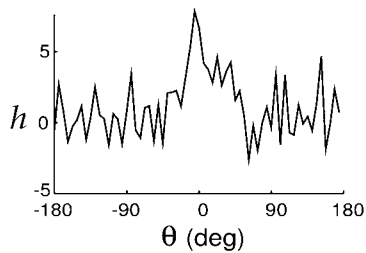
Choose recurrent connections =  
cosine function of relative angle

Excitation nearby,  
Inhibition further away

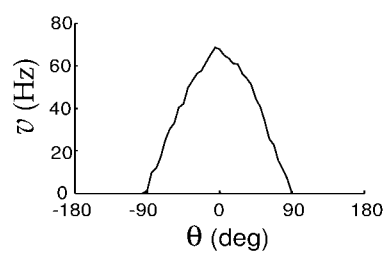


## Amplification in a Nonlinear Recurrent Network

Input

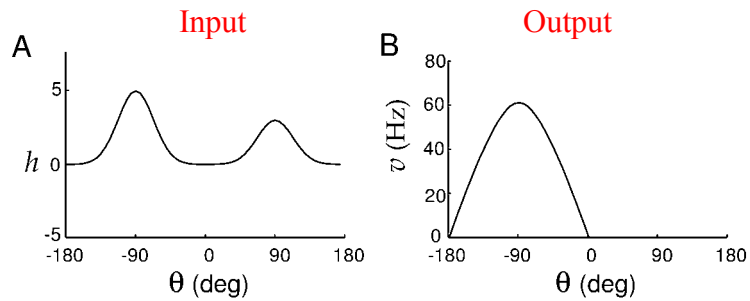


Output



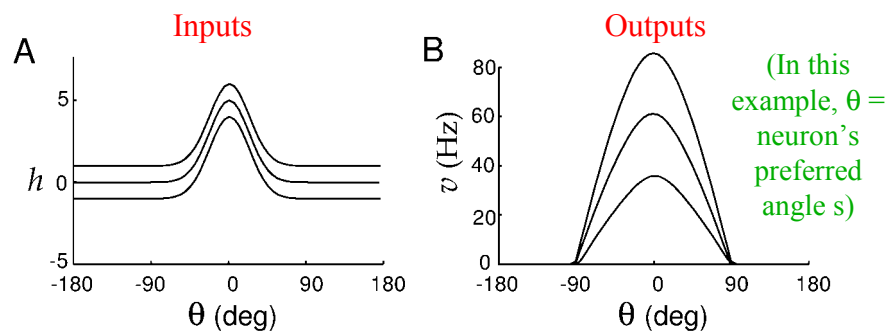
$$\tau \frac{dv(\theta)}{dt} = -v(\theta) + \left[ h(\theta) + \frac{\lambda_1}{\pi} \int_{-\pi}^{\pi} \cos(\theta - \theta') v(\theta') d\theta' \right]^+$$

## Selective “Attention” in a Nonlinear Recurrent Network



Network performs “winner-takes-all” input selection

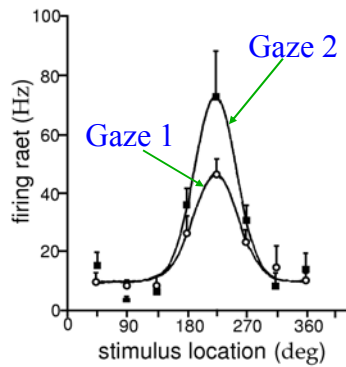
## Gain Modulation in a Nonlinear Recurrent Network



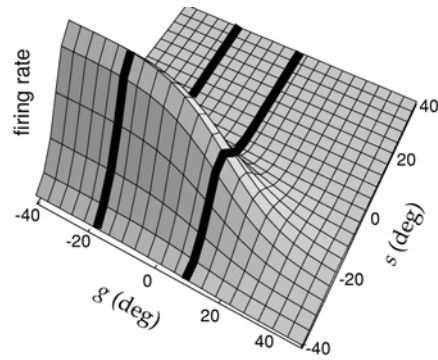
Changing the level of input by adding  $g$  multiplies the output

If  $h = s + g$  ( $s =$  stimulus angle on retina,  $g =$  gaze angle), then network output is gain-modulated similar to parietal cortex neurons

## Gain Modulation in Parietal Cortex Neurons

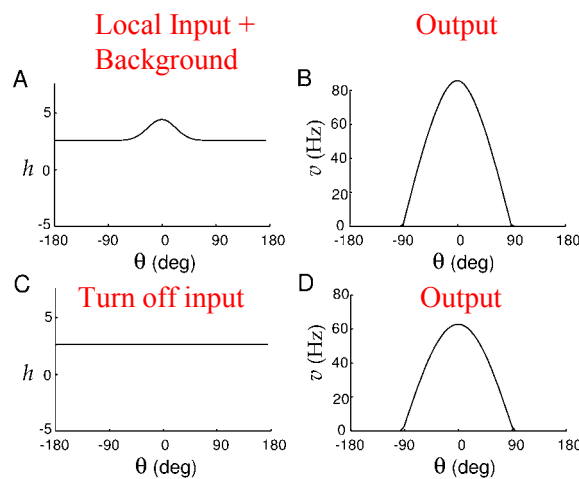


Responses of Area 7a neuron



Example of a gain-modulated tuning curve

## Short-Term Memory Storage in a Nonlinear Recurrent Network

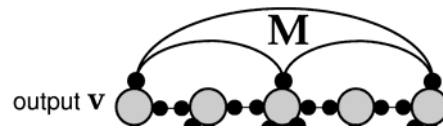


Network maintains a *memory of previous activity* when input is taken away.

Similar to “short-term memory” or “working memory” in prefrontal cortex)

## Associative Memories (Hopfield Networks)

- ◆ Fully connected, no feedforward inputs



**Idea:** Store patterns as *fixed points* of this network

$$\tau \frac{d\mathbf{I}}{dt} = -\mathbf{I} + \mathbf{M} \cdot \mathbf{g}(\mathbf{I}) \quad \text{or,}$$

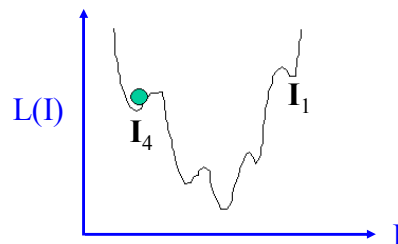
$$\tau \frac{dI_i}{dt} = -I_i + \sum_j M_{ij} v_j \quad \text{where } v_j = g(I_j)$$

$g = \text{sigmoid function}$

**Question:** Will  $\mathbf{I}$  always converge to fixed point?

## Enter...Lyapunov Functions

- ◆ **Idea:** If  $d\mathbf{I}/dt$  causes some function  $L(\mathbf{I})$  to always decrease or remain constant (i.e.  $dL/dt \leq 0$ ) and  $L$  has a lower bound (with  $dL/dt = 0$  only if  $d\mathbf{I}/dt = 0$ ), *then  $d\mathbf{I}/dt = 0$  eventually*
  - ⇨ **Network converges to a fixed point**
- ◆  $L$  also called “energy” function or “cost” function





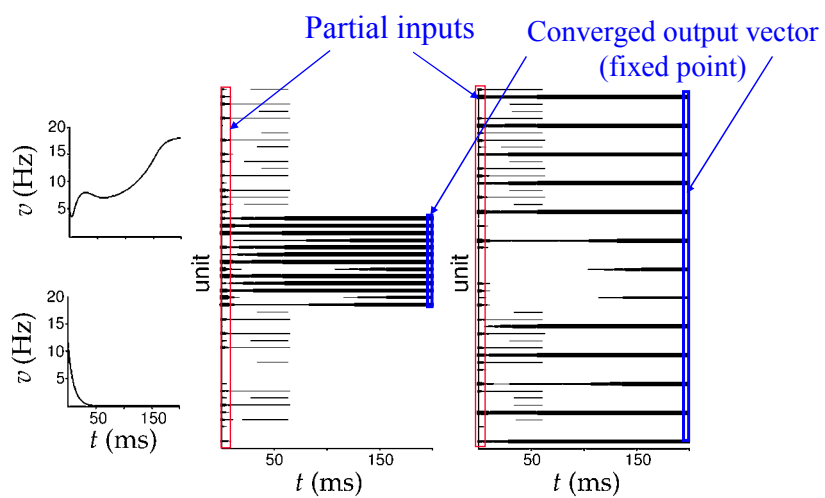
## Lyapunov for Hopfield networks

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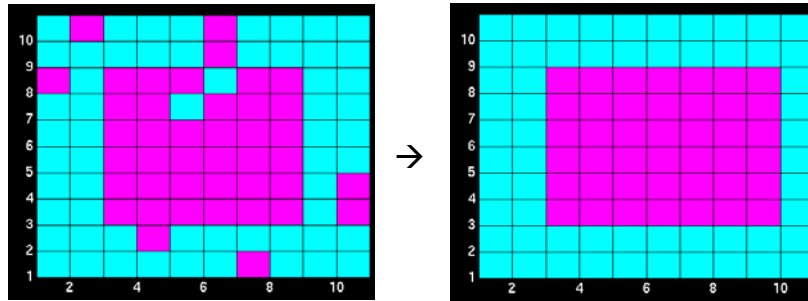
- ◆ What is a good Lyapunov function  $L(I)$  for Hopfield nets?
- ◆ What constraints are required on the recurrent weights  $\mathbf{M}$ ?
- ◆ On-board example:  $L(I)$

## Example of Auto-Associative Memory

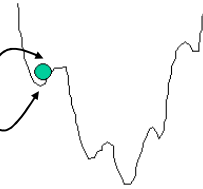
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## Pattern Completion in a Hopfield Network



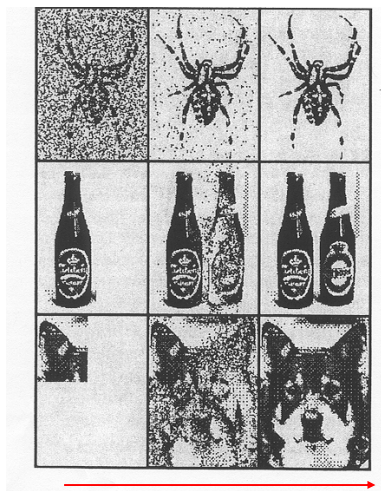
Network converges  
from here  
to here



Local minimum  
("attractor") of cost  
(or "energy") function  
stores pattern

## Pattern Recall in Hopfield Nets

Initial states



Stable states  
(fixed points)

## Non-Symmetric Recurrent Networks

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- ◆ Example: Network of Excitatory (E) and Inhibitory (I) Neurons

⇒ Connections can't be symmetric: Why?

$$\tau_E \frac{dv_E}{dt} = -v_E + [M_{EE}v_E + M_{EI}v_I - \gamma_E]^+$$

$$\tau_I \frac{dv_I}{dt} = -v_I + [M_{II}v_I + M_{IE}v_E - \gamma_I]^+$$

Simple 2 neuron model for representing interacting populations  
One excitatory neuron and one inhibitory neuron

## Non-Symmetric Recurrent Networks

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- ◆ Example: Network of Excitatory (E) and Inhibitory (I) Neurons

⇒ Connections can't be symmetric:  $M_{EI} < 0$  and  $M_{IE} > 0$

10 ms

$$\tau_E \frac{dv_E}{dt} = -v_E + [M_{EE}v_E + M_{EI}v_I - \gamma_E]^+$$

$$\tau_I \frac{dv_I}{dt} = -v_I + [M_{II}v_I + M_{IE}v_E - \gamma_I]^+$$

Parameter

1.25    -1    -10

0    1    10

## Linear Stability Analysis

$$\frac{dv_E}{dt} = \frac{-v_E + [M_{EE}v_E + M_{EI}v_I - \gamma_E]}{\tau_E}$$

Take derivatives of right hand side with respect to both  $v_E$  and  $v_I$

$$\frac{dv_I}{dt} = \frac{-v_I + [M_{II}v_I + M_{IE}v_E - \gamma_I]}{\tau_I}$$

◆ Matrix of derivatives (the “Jacobian Matrix”):

$$J = \begin{bmatrix} \frac{(M_{EE} - 1)}{\tau_E} & \frac{M_{EI}}{\tau_E} \\ \frac{M_{IE}}{\tau_I} & \frac{(M_{II} - 1)}{\tau_I} \end{bmatrix}$$

## Compute the Eigenvalues

◆ Jacobian Matrix:

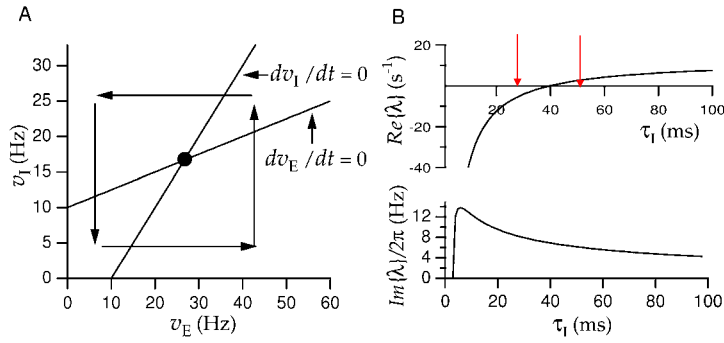
$$J = \begin{bmatrix} \frac{(M_{EE} - 1)}{\tau_E} & \frac{M_{EI}}{\tau_E} \\ \frac{M_{IE}}{\tau_I} & \frac{(M_{II} - 1)}{\tau_I} \end{bmatrix}$$

◆ Its two eigenvalues (obtained by solving  $\det(J - \lambda I) = 0$ ):

$$\lambda = \frac{1}{2} \left( \frac{(M_{EE} - 1)}{\tau_E} + \frac{(M_{II} - 1)}{\tau_I} \pm \sqrt{\left( \frac{(M_{EE} - 1)}{\tau_E} - \frac{(M_{II} - 1)}{\tau_I} \right)^2 + 4 \frac{M_{EI}M_{IE}}{\tau_E\tau_I}} \right)$$

Different dynamics depending on real and imaginary parts of  $\lambda$   
(see pages 410-412 of Appendix in Text)

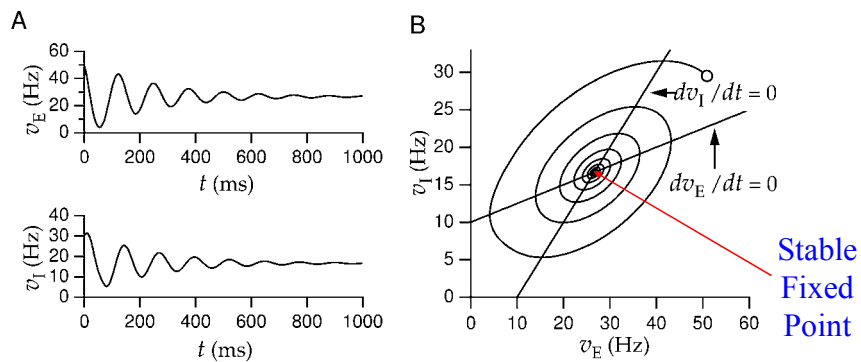
## Phase Plane and Eigenvalue Analysis



$$\tau_E \frac{dv_E}{dt} = -v_E + [1.25v_E - v_I + 10]^+$$

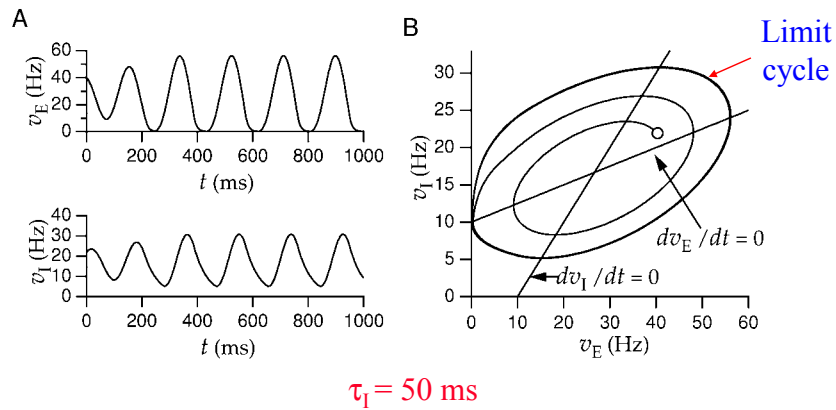
$$\tau_I \frac{dv_I}{dt} = -v_I + [0 \cdot v_I + v_E - 10]^+$$

## Damped Oscillations in the Network

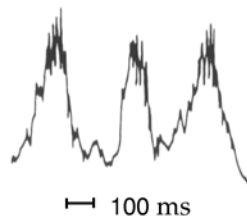


$$\tau_I = 30 \text{ ms}$$

## Unstable Behavior and Limit Cycle



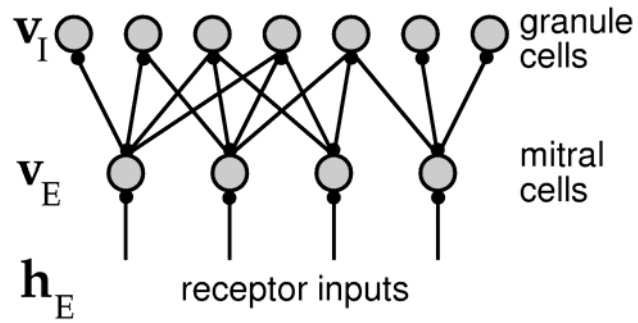
## Oscillatory Activity in Real Networks



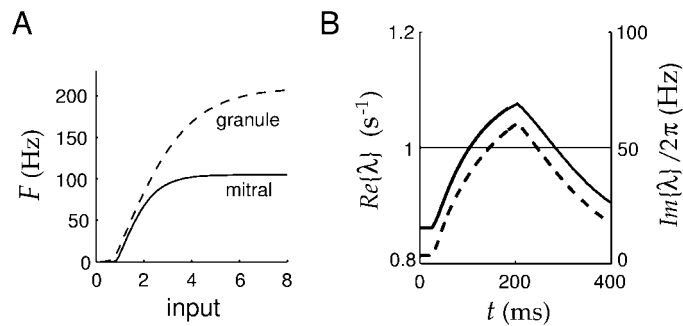
Activity in rabbit (or wabbit) olfactory bulb during 3 sniffs



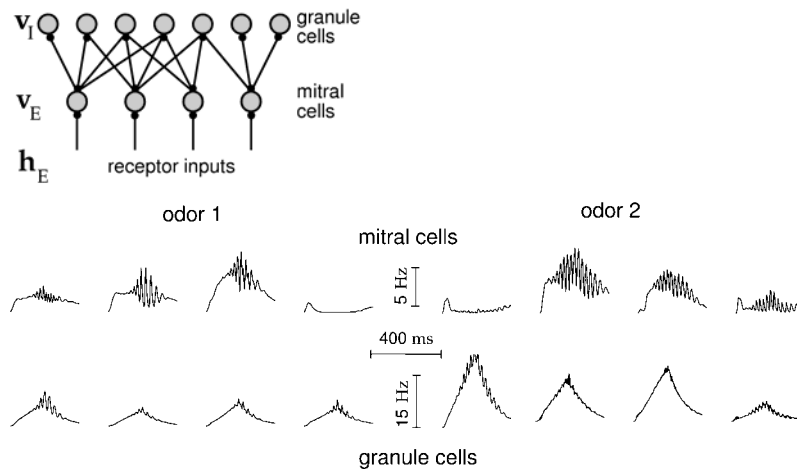
## Olfactory Bulb Network



## Analysis of Olfactory Bulb Oscillations



## Output of Model Network for Two Odors



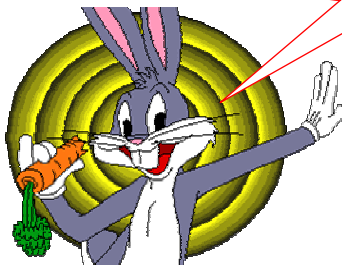
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## Next Class: Plasticity and Learning

- ◆ Things to do:
  - ⇒ Start reading Chapter 8
  - ⇒ Homework #3 due by Fri 5pm
  - ⇒ Start on mini-project

That's all folks!  
Have a great  
weekend.



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