Semantics of FO Formulas

Semantics

- Given a vocabulary $R_1, \ldots, R_k$
- A *model* is $\mathbf{D} = (D, R_1^D, \ldots, R_k^D)$
  - D = a set, called domain, or universe
  - $R_i^D \subseteq D \times D \times \ldots \times D$, (ar($R_i$) times) i = 1,...,k

Semantics

- Given:
  - A model $\mathbf{D} = (D, R_1^D, ..., R_k^D)$
  - A formula φ
  - A substitution $s : \{x_1, x_2, ...\} \rightarrow D$
- We define next the relation:

$\mathbf{D} \models \varphi[s]$

meaning "D satisfies with s"

Semantics

$\mathbf{D} \models (R(t_1, ..., t_n))\ [s]$ If $(s(t_1), ..., s(t_n)) \in R^D$

$\mathbf{D} \models (t = t')\ [s]$ If $s(t) = s(t')$

Semantics

$\mathbf{D} \models (\varphi \wedge \varphi')\ [s]$ If $\mathbf{D} \models (\varphi)[s]$ and $\mathbf{D} \models (\varphi')\ [s]$

$\mathbf{D} \models (\varphi \vee \varphi')\ [s]$ If $\mathbf{D} \models (\varphi)[s]$ or $\mathbf{D} \models (\varphi')\ [s]$

$\mathbf{D} \models (\neg\varphi')\ [s]$ If not $\mathbf{D} \models (\varphi)[s]$

First Order Logic: Semantics

$\mathbf{D} \models (\forall x.\varphi)\ [s]$ If for all s' s.t. s(y) = s'(y) for all variables y other than x, $\mathbf{D} \models (\varphi)[s']$

$\mathbf{D} \models (\exists x.\varphi)\ [s]$ If for some s' s.t. s(y) = s'(y) for all variables y other than x, $\mathbf{D} \models (\varphi)[s']$

FO and Databases

- FO:
  a sentence φ is *true* in **D** if **D** |= φ
- Databases:
  a formula φ with free variables x_1, ..., x_n
  defines the query:
  $\varphi(\mathbf{D}) = \{(s(x_1), ..., s(x_n)) \mid \mathbf{D} \models \varphi[s]\}$