Computer Graphics	**Prof. Brian Curless**
CSE 557	**Winter 2004**

Homework #1

Sampling theory, image processing, affine transformations

Assigned: Sunday, January 25, 2004
Due: Thursday, February 5, 2004

Directions: Please provide short written answers to the following questions. For problem #1, use and attach additional sheets of paper. You can use the space provided for the other problems. Feel free to talk over the problems in general terms with classmates, but please *answer the questions on your own*.

Name:____

1. Fourier transforms and signal reconstruction
62 Points

In this problem, you will take a closer look at Fourier transforms and signal reconstruction. We begin with a couple of preliminaries. Recall that the box function II(x) is defined as:

$$\mathrm{II}(x) = \begin{cases} 1 & |x| < 1/2 \\ 1/2 & |x| = 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

and its Fourier transform is sinc(s) = sin(πs)/πs. Likewise, the Fourier transform of sinc(x) is II(s). In either case, we can show that sinc(0) = 1.

The hat function, $\Lambda(x)$, is defined as:

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

We sample a function by multiplying with the comb or shah function:

$$\hat{f}(x) = f(x)\mathrm{III}(x) = \sum_{i=-\infty}^{i=\infty} f(i)\delta(x-i)$$

We reconstruct by convolving with a reconstruction filter $r(x)$:

$$\tilde{f}(x) = r(x) * \hat{f}(x) = \sum_{i=-\infty}^{i=\infty} f(i)r(x-i)$$

The last two equations generalize naturally into 2D. One more convenient fact is that separable filters of the form $c(x,y) = a(x)b(y)$ lead to simplified convolution:

$$\begin{aligned} g(x,y) *_{\mathrm{2D}} c(x,y) &= \iint g(x',y')c(x-x',y-y')dx'dy' \\ &= \iint g(x',y')a(x-x')b(y-y')dx'dy' \\ &= \int \left\{ \int g(x',y')a(x-x')dx' \right\} b(y-y')dy' \\ &= g(x,y) *_{\mathrm{1D}} a(x) *_{\mathrm{1D}} b(y) \end{aligned}$$

So, you can convolve your signal first in the x-direction with $a(x)$, and then in the y-direction with $b(y)$. You can also convolve first in y then x; the answer is the same in either case.

Problem 1 (cont'd.)

Now we get to the problems you need to solve:

a) The "ideal" reconstruction filter is r(x) = sinc(x). Is this filter an interpolating filter, i.e., will the reconstructed function $\tilde{f}(x)$ pass through the original sample points $f(j)$? In other words, do we expect to find that $\tilde{f}(j) = f(j)$, for all integer j? Justify your answer.

b) Let's say we reconstruct with the sinc(x) function, and we want to evaluate $\tilde{f}(1/2)$. Assume that the samples $f(i)$ are positive 8-bit numbers (0-255), that we perform the reconstruction using floating point math (including negative numbers), and that we want an accurate reconstruction such that the smallest contribution from *any* original sample to the reconstruction summation can be at most ½ bit, i.e, +/-0.5 on a scale of 0-255. Given no prior knowledge about $f(x)$, how many samples $f(i)$ must, in the worst case, be included to compute $\tilde{f}(1/2)$?

c) If we use $\Lambda(x)$ as a reconstruction filter, what is the equation for computing $\tilde{f}(x)$ where we decompose the domain variable into $x = j + \Delta$, and j = floor(x), and Δ is a fractional value between 0 and 1?

d) Is $\Lambda(x)$ an interpolating filter? How many samples are needed to compute $\tilde{f}(1/2)$ to the same precision as in (b)?

e) It is easy to show that $\Lambda(x) = \Pi(x)*\Pi(x)$. (Try integrating it yourself, or use the convolution applet.) What is the Fourier transform of $\Lambda(x)$? Sketch the function.

f) One of the most common 2D resampling methods used in graphics is *bilinear interpolation*, defined mathematically as using a reconstruction filter $r(x, y) = \Lambda(x)\Lambda(y)$. Suppose we now have a 2D input function $f(x,y)$ that has been sampled. What is the equation for $\tilde{f}(x, y)$ where we decompose the domain variables into $x = j + \Delta x$, $y = k + \Delta y$, j = floor(x), k = floor(y), and Δx and Δy each range between 0 and 1?

g) Sketch the shape of $\Lambda(x)\Lambda(y)$. Note: there is a partial sketch of this in the notes, but it really only shows the "footprint" and the x- and y-axis cross-sections. Your sketch should give a better idea of the rest of its shape.

h) The "ideal" alternative to using bilinear interpolation is to use the function sinc(x)sinc(y). Suppose you were wanted to compute $\tilde{f}(1/2, 1/2)$. Based on your answers to (b) and (f), estimate the number of input samples needed to compute this function value for both bilinear interpolation and sinc(x)sinc(y) reconstruction.

i) What is the Fourier transform of the filter $r(x, y) = \Lambda(x)\Lambda(y)$? You will need to manipulate the equation for the 2D Fourier transform to justify your answer, but you will not need to compute any integrals. You do not need to sketch the function.

2. Image processing
11 points

Describe the effect of each of the following filters. In addition, indicate which filter will cause the most blurring and which, when convolved with a solid (positive) intensity image, will produce the brightest image and which will produce the darkest image. Justify your answers.

0.1	0.1	0.1
0.1	0.1	0.1
0.1	0.1	0.1

0	0	1
0	-2	0
1	0	0

0	0.2	0
0.2	0.4	0.2
0	0.2	0

0	-1	0
0	3	0
0	-1	0

3. Affine Transformations
27 points

$$A=\begin{bmatrix}1&0&0&6\\0&1&0&3\\0&0&1&2\\0&0&0&1\end{bmatrix}\quad B=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&1\end{bmatrix}\quad C=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$D=\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}\quad E=\begin{bmatrix}2&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}\quad F=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$G=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\quad H=\begin{bmatrix}1&0&-1&0\\0&1&0&0\\1&0&1&0\\0&0&0&1\end{bmatrix}\quad I=\begin{bmatrix}1&0&0&1\\0&.6&.8&0\\0&-.8&.6&0\\0&0&0&1\end{bmatrix}$$

a) As discussed in class, any three-dimensional affine transformation can be represented with a 4x4 matrix. Match each of the matrices above to exactly one of the following transformations (not all blanks will be filled):

____ Differential (Non-Uniform) Scaling

____ Reflection

____ Rotation about the z-axis with non-uniform scaling

____ Rotation about the y-axis with non-uniform scaling

____ Translation

____ Rotation about the x-axis

____ Rotation about the y-axis

____ Rotation about the z-axis

____ Shearing along *z* with respect to the x-y plane (z=0 plane unchanged by shear)

____ Shearing along *x* with respect to the y-z plane (x=0 plane unchanged by shear)

____ Rotation about the x-axis and translation

____ Uniform scaling

____ Reflection with uniform scaling

Problem 3 (cont'd.)

b) Consider a line that passes through a point **p** = (p_x, p_y, p_z) in the direction **v** = (cos α, 0, sin α). Write out the product of matrices that would perform a rotation by θ about this line. You should **not** multiply these matrices out, but you do need to write out all of the elements in these matrices.