Subdivision curves and surfaces

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CSE 557
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Reading

Required:

- Stollnitz, DeRose, and Salesin. *Wavelets for Computer Graphics: Theory and Applications,* 1996, section 6.1-6.3, 10.2, A.5.

Note: there is an error in Stollnitz, et al., section A.5. Equation A.3 should read:

$$\mathbf{MV} = \mathbf{V}\Lambda$$

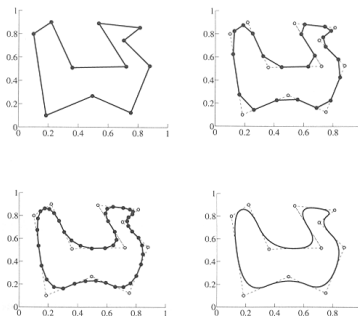This is already fixed in the handout.

Subdivision curves

Idea:

- repeatedly refine the control polygon

$$P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow \cdots$$

- curve is the limit of an infinite process
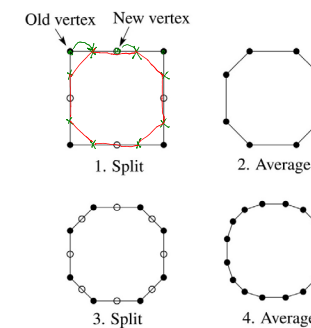
$$Q = \lim_{j \rightarrow \infty} P_j$$

Chaikin's algorithm

Chakin introduced the following "corner-cutting" scheme in 1974:

- Start with a piecewise linear curve
- Insert new vertices at the midpoints (the **splitting step**)
- Average each vertex with the "next" (clockwise) neighbor (the **averaging step**)
- Go to the splitting step

Old vertex   New vertex

1. Split   2. Average

3. Split   4. Average

Averaging masks

The limit curve is a quadratic B-spline!

Instead of averaging with the nearest neighbor, we can generalize by applying an **averaging mask** during the averaging step:

$$r = \begin{bmatrix} \ldots & r_{-1} & r_0 & r_1 & \end{bmatrix}$$

In the case of Chaikin's algorithm:

$$r = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
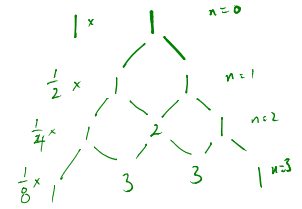
$$= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Lane-Riesenfeld algorithm (1980)

Use averaging masks from Pascal's triangle:

$$r = \frac{1}{2^n} \begin{bmatrix} \binom{n}{0} & \binom{n}{1} & \ldots & \binom{n}{n} \end{bmatrix}$$

Gives B-splines of degree $n+1$.

n=0: $\begin{bmatrix} 1 \end{bmatrix}$ linear

n=1: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ quadratic

n=2: $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$ cubic

Subdivide ad nauseum?

After each split-average step, we are closer to the **limit curve**.

How many steps until we reach the final (limit) position?
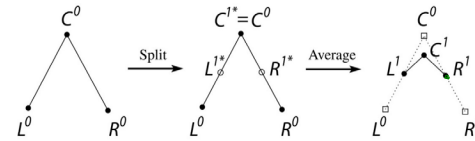
Can we push a vertex to its limit position in one step?

Local subdivision matrix

Consider the cubic B-spline subdivision mask:

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Now consider what happens during splitting and averaging in a small neighborhood:

We can write equations that relate points at one subdivision level to points at the previous:

$$L^{1*} = \frac{1}{2}L^0 + \frac{1}{2}C^0 \qquad C^{1*} = C^0 \qquad R^{1*} = \frac{1}{2}C^0 + \frac{1}{2}R^0$$

$$L^1 = \frac{1}{4}L^0 + \frac{1}{2}L^{1*} + \frac{1}{4}C^0$$

$$= \frac{1}{4}L^0 + \frac{1}{4}L^0 + \frac{1}{4}C^0 + \frac{1}{4}C^0$$

$$= \frac{1}{2}L^0 + \frac{1}{2}C^0$$

$$R^1 = \frac{1}{2}C^0 + \frac{1}{2}R^0$$

$$C^1 = \frac{1}{4}L^{1*} + \frac{1}{2}C^0 + \frac{1}{4}R^{1*}$$

$$= \frac{1}{4}\left(\frac{1}{2}L^0 + \frac{1}{2}C^0\right) + \frac{1}{2}C^0 + \frac{1}{4}\left(\frac{1}{2}C^0 + \frac{1}{2}R^0\right)$$

$$= \frac{1}{8}L^0 + \frac{3}{4}C^0 + \frac{1}{8}R^0$$

Local subdivision matrix

We can write this as a recurrence relation in matrix form:

$$\begin{bmatrix} L_j \\ C_j \\ R_j \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/8 & 3/4 & 1/8 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} L_{j-1} \\ C_{j-1} \\ R_{j-1} \end{bmatrix}$$

where the *L, R, C*'s are (for convenience) *row* vectors.

In 2D, we can write out all the elements as follows:

$$\underbrace{\begin{bmatrix} L_j^x & L_j^y & 1 \\ C_j^x & C_j^y & 1 \\ R_j^x & R_j^y & 1 \end{bmatrix}}_{A_j} = \underbrace{\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/8 & 3/4 & 1/8 \\ 0 & 1/2 & 1/2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} L_{j-1}^x & L_{j-1}^y & 1 \\ C_{j-1}^x & C_{j-1}^y & 1 \\ R_{j-1}^x & R_{j-1}^y & 1 \end{bmatrix}}_{A_{j-1}}$$

We can re-write this as:

$$\mathbf{A}_j = \mathbf{M}\mathbf{A}_{j-1}$$

and **M** is the **local subdivision matrix**.

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Local subdivision matrix, cont'd

Starting from the initial control polygon, we can track the original vertex and its original neighborhood through subdivision:

$$\mathbf{A}_j = \mathbf{M}\mathbf{A}_{j-1} = M\left(MA_{j-2}\right) = M\left(M(MA_{j-3})\right) = \cdots = M^j A_0$$

$$M^2 A_{j-2} \qquad M^3 A_{j-3}$$

The limit position of the neighborhood is then:

$$\mathbf{A}_\infty = \lim_{j \to \infty} \mathbf{M}^j \mathbf{A}_0$$

OK, so how do we apply a matrix an infinite number of times??

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Eigenvectors and eigenvalues

We now need to look at the eigenvectors and eigenvalues of **M**. Let **v** be a vector such that:

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$$

We say that **v** is an eigenvector of **M** with eigenvalue λ.

A 3x3 matrix can have 3 eigenvalues and eigenvectors:

$$\mathbf{M}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$$
$$\mathbf{M}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$$
$$\mathbf{M}\mathbf{v}_3 = \lambda_3\mathbf{v}_3$$

In matrix form:

$$\mathbf{M}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \lambda_3\mathbf{v}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\mathbf{MV} = \mathbf{V}\Lambda$$

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To infinity, but not beyond...

$$\Lambda^2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Now let's apply **M** to original neighborhood $\mathbf{A}_0$:

$$\mathbf{A}_1 = \mathbf{M}\mathbf{A}_0 = M V V^{-1} A_0 = V \Lambda V^{-1} A_0 \qquad = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

Now let's advance another subdivision:

$$\mathbf{A}_2 = \mathbf{M}^2\mathbf{A}_0 = \mathbf{M}\mathbf{A}_1 = M V \Lambda V^{-1} A_0 = V \Lambda \Lambda V^{-1} A_0 = V \Lambda^2 V^{-1} A_0$$

Do it j times:

$$\mathbf{A}_j = \mathbf{M}^j\mathbf{A}_0 = V \Lambda^j V^{-1} A_0 = V \begin{bmatrix} \lambda_1^j & 0 & 0 \\ 0 & \lambda_2^j & 0 \\ 0 & 0 & \lambda_3^j \end{bmatrix} V^{-1} A_0$$

What if we do this an infinite number of times?

$$\mathbf{A}_\infty = \mathbf{M}^\infty\mathbf{A}_0 = V \begin{bmatrix} \lambda_1^\infty & 0 & 0 \\ 0 & \lambda_2^\infty & 0 \\ 0 & 0 & \lambda_3^\infty \end{bmatrix} V^{-1} A_0$$

Let's assume the eigenvalues are non-negative and sorted so that:

$$\lambda_1 > \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_n \geq 0$$

If $\lambda_1 > 1$, then: blows up

If $\lambda_1 < 1$, then: implodes

If $\lambda_1 = 1$, then: stable

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Evaluation masks

For cubic B-splines, the local subdivision matrix **M** is:

$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/8 & 3/4 & 1/8 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

It's eigenvalues and eigenvectors are:

$$\lambda_1 = 1 \qquad \lambda_2 = \frac{1}{2} \qquad \lambda_3 = \frac{1}{4}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$\lambda_1 = 1 > \lambda_2 > \lambda_3$, so we're OK!

We can write out Λ and **V**:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

We will also need $\mathbf{V}^{-1}$, which turns out to be:

$$\mathbf{V}^{-1} = \begin{bmatrix} 1/6 & 2/3 & 1/6 \\ -1/2 & 0 & 1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

Evaluation masks (cont'd)

So, we have:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \quad \mathbf{V}^{-1} = \begin{bmatrix} 1/6 & 2/3 & 1/6 \\ -1/2 & 0 & 1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

We can now compute the limit position of the neighborhood $\mathbf{A}_0$:

$$\mathbf{A}_\infty = \mathbf{M}^\infty \mathbf{A}_0 = \mathbf{V}\Lambda^\infty \mathbf{V}^{-1}\mathbf{A}_0$$

$$d_{center}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$d_{tangent}^T = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/6 & 2/3 & 1/6 \\ -1/2 & 0 & 1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix} A_0$$

$$d_{2nd\ deriv}^T = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/6 & 2/3 & 1/6 \\ -1/2 & 0 & 1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix} A_0$$

$$= \begin{bmatrix} 1/6 & 2/3 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1/6 & 2/3 & 1/6 \end{bmatrix} \begin{bmatrix} L^0 \\ C^0 \\ R^0 \end{bmatrix}$$

$$C^\infty = \frac{1}{6}L^0 + \frac{2}{3}C^0 + \frac{1}{6}R^0$$

Recipe for subdivision curves

The row vector $\mathbf{u}_1^T$ that pushes the original vertex to the limit position is called the **evaluation mask**:

$$\mathbf{u}_1^T = \begin{bmatrix} \dfrac{1}{6} & \dfrac{2}{3} & \dfrac{1}{6} \end{bmatrix}$$

Note that we do **not** need start with the 0th level control points and push them to the limit.

If we subdivide and average the control polygon j times, we can push the vertices of the refined polygon to the limit as well:

$$\mathbf{A}_\infty = \mathbf{M}^\infty \mathbf{A}_j = \mathbf{u}_1^T \mathbf{A}_j$$

Now we can cook up a simple procedure for creating subdivision curves:

- Subdivide (split+average) the control polygon a few times. Use the averaging mask.
- Push the resulting points to the limit positions. Use the evaluation mask.

$$U = V^{-1} = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}$$

$$MV = V\Lambda$$

$$V^{-1}MV = V^{-1}V\Lambda$$

$$V^{-1}MV = \Lambda$$

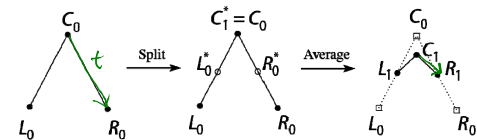$$V^{-1}MVV^{-1} = \Lambda V^{-1}$$

$$V^{-1}M = \Lambda V^{-1}$$

$$UM = \Lambda U$$

$$\begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} M = \begin{bmatrix} \lambda_1 \\ & \lambda_2 \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}$$

$$u_1^T M = \lambda_1 u_1^T$$

Tangent analysis

$$A_j = \begin{bmatrix} L_j^x & L_j^y & 1 \\ C_j^x & C_j^y & 1 \\ R_j^x & R_j^y & 1 \end{bmatrix}$$

$$t_j = R_j - C_j = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} L_j^x & L_j^y & 1 \\ C_j^x & C_j^y & 1 \\ R_j^x & R_j^y & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} A_j = d^T A_j$$

$$d = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$t_j = d^T A_j = d^T V \Lambda^j V^{-1} A_0 = d^T \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \Lambda^j V^{-1} A_0$$

$$= \begin{bmatrix} \underbrace{d^T v_1}_{w_1} & \underbrace{d^T v_2}_{w_2} & \underbrace{d^T v_3}_{w_3} \end{bmatrix} \begin{bmatrix} \lambda_1^j \\ & \lambda_2^j \\ & & \lambda_3^j \end{bmatrix} \underbrace{V^{-1} A_0}_{U}$$

$$w_1 = d^T v_1 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^j w_1 & \lambda_2^j w_2 & \lambda_3^j w_3 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} A_0$$

$$= 0 \qquad = \left(\lambda_1^j w_1 u_1^T + \lambda_2^j w_2 u_2^T + \lambda_3^j w_3 u_3^T \right) A_0$$

Tangent analysis (cont'd)

$$\mathbf{t}_j = \left(\lambda_1^j w_1 \mathbf{u}_1^T + \lambda_2^j w_2 \mathbf{u}_2^T + \lambda_3^j w_3 \mathbf{u}_3^T\right) A_0 \qquad \lambda_1 > \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_n \geq 0$$

$$\hat{t}_j = \frac{\left(\lambda_2^j w_2 u_2^T + \lambda_3^j w_3 u_3^T\right) A_0}{\| \lambda_2^j w_2 u_2^T + \lambda_3^j w_3 u_3^T) A_0 \|} \cdot \frac{1/\lambda_2^j}{1/\lambda_2^j}$$

$$= \frac{\left(w_2 u_2^T + (\lambda_3/\lambda_2)^j w_3 u_3^T\right) A_0}{\|(w_2 u_2^T + (\lambda_3/\lambda_2)^j w_3 u_3^T) A_0 \|} \qquad \frac{\lambda_3}{\lambda_2} < 1$$

$$\lim_{j \to \infty} \hat{t}_j = \frac{w_2 u_2^T A_0}{\|(w_2 u_2^T A_0 \|}$$

$$= \frac{u_2^T A_0}{\| u_2^T A_0 \|}$$

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DLG interpolating scheme (1987)

Slight modification to subdivision algorithm:

- ◆ splitting step introduces midpoints
- ◆ averaging step *only changes midpoints*
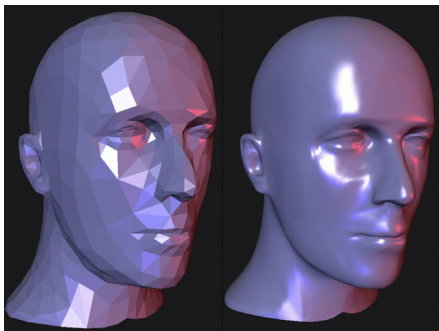
For DLG (Dyn-Levin-Gregory), use:

$$r_{old} = (1) \qquad r_{new} = \frac{1}{16}(-2, 5, 10, 5, -2)$$
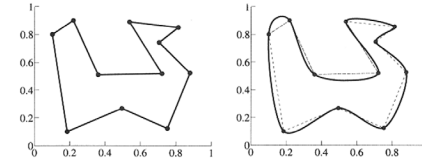
Since we are only changing the midpoints, the points after the averaging step do not move.

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Building complex models

We can extend the idea of subdivision from curves to surfaces…
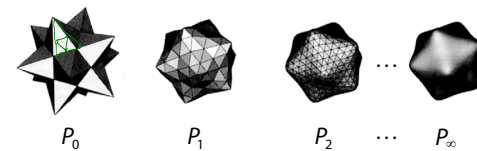

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Subdivision surfaces

Chaikin's use of subdivision for curves inspired similar techniques for subdivision surfaces.

Iteratively refine a **control polyhedron** (or **control mesh**) to produce the limit surface
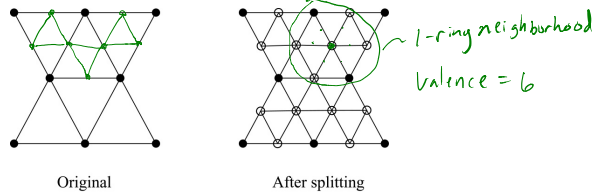
$$S = \lim_{j \to \infty} P_j$$

using splitting and averaging steps.

$P_0 \qquad P_1 \qquad P_2 \qquad \cdots \qquad P_\infty$

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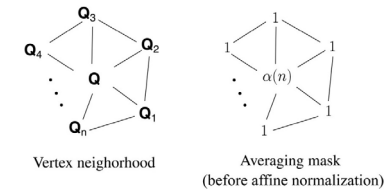## Triangular subdivision

There are a variety of ways to subdivide a poylgon mesh.

A common choice for triangle meshes is 4:1 subdivision – each triangular face is split into four smaller triangles:

1-ring neighborhood
Valence = 6

Original   After splitting

Loop averaging step

Once again we can use **masks** for the averaging step:

Vertex neighborhood   Averaging mask
(before affine normalization)

$$\mathbf{Q} \leftarrow \frac{\alpha(n)\mathbf{Q} + \mathbf{Q}_1 + \cdots + \mathbf{Q}_n}{\alpha(n) + n}$$

where

$$\alpha(n) = \frac{n(1-\beta(n))}{\beta(n)} \quad \beta(n) = \frac{5}{4} - \frac{(3+2\cos(2\pi/n))^2}{32}$$
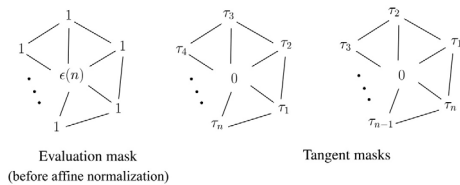
These values, due to Charles Loop, are carefully chosen to ensure smoothness – namely, tangent plane or normal continuity.

Note: tangent plane continuity is also know as G^1 continuity for surfaces.

Loop evaluation and tangent masks

Curves
$$\lambda_1 > \lambda_2 > \lambda_3 \cdots \geq 0$$

As with subdivision curves, we can split and average a number of times and then push the points to their limit positions.

Surfaces
$$\lambda_1 > \lambda_2 = \lambda_3 > \cdots$$

Evaluation mask
(before affine normalization)   Tangent masks

$$\mathbf{Q}^\infty = \frac{\varepsilon(n)\mathbf{Q} + \mathbf{Q}_1 + \cdots + \mathbf{Q}_n}{\varepsilon(n) + n}$$

$$\mathbf{T}_1^\infty = \tau_1(n)\mathbf{Q}_1 + \tau_2(n)\mathbf{Q}_2 + \cdots + \tau_n(n)\mathbf{Q}_n$$

$$\mathbf{T}_2^\infty = \tau_n(n)\mathbf{Q}_1 + \tau_1(n)\mathbf{Q}_2 + \cdots + \tau_{n-1}(n)\mathbf{Q}_n$$

where

$$\varepsilon(n) = \frac{3n}{\beta(n)} \quad \tau_i(n) = \cos(2\pi i/n)$$

How do we compute the normal?

$$\mathbf{T}_1^\infty \times \mathbf{T}_2^\infty$$

Recipe for subdivision surfaces

As with subdivision curves, we can now describe a recipe for creating and rendering subdivision surfaces:

- ◆ Subdivide (split+average) the control polyhedron a few times. Use the averaging mask.
- ◆ Compute two tangent vectors using the tangent masks.
- ◆ Compute the normal from the tangent vectors.
- ◆ Push the resulting points to the limit positions. Use the evaluation mask.
- ◆ Render!

Adding creases without trim curves

For NURBS surfaces, adding sharp features like creases required the use of trim curves.

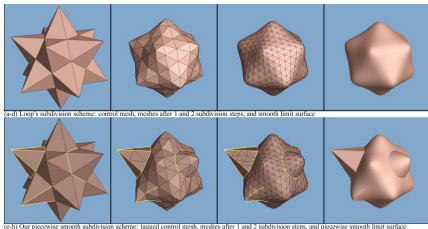$$\begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

For subdivision surfaces, we can just modify the subdivision masks. E.g., we can mark some edges and vertices as "creases" and modify the subdivision mask for them (and their children):
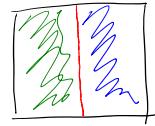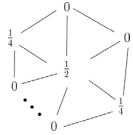
$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

This gives rise to G^0 continuous surfaces (i.e., having positional but not tangent plane continuity).


[Hoppe, SIGGRAPH 1994]

(a-d) Loop's subdivision scheme: control mesh, meshes after 1 and 2 subdivision steps, and smooth limit surface

(e-h) Our piecewise smooth subdivision scheme: tagged control mesh, meshes after 1 and 2 subdivision steps, and piecewise smooth limit surface
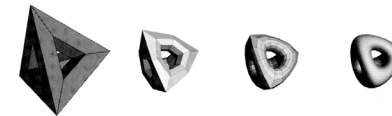

Catmull-Clark subdivision

4:1 subdivision of triangles is sometimes called a **face scheme** for subdivision, as each face begets more faces.

An alternative face scheme starts with arbitrary polygon meshes and inserts vertices along edges and at face centroids:

Original   After splitting
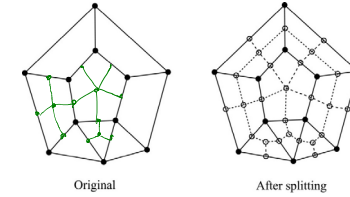
Catmull-Clark subdivision:

Note: after the first subdivision, all polygons are quadilaterals in this scheme.

Creases without trim curves, cont.

Here's an example using Catmull-Clark surfaces (based on subdividing quadrilateral meshes):


This particular example uses the hybrid technique of DeRose, et al., which applies sharp subdivision rules at some creases for a finite number of steps, and then switches to smooth subdivision, giving more gentle creases. This technique was used in Geri's Game.

Interpolating subdivision surfaces

Interpolating schemes are defined by

- splitting
- averaging only new vertices

The following averaging mask is used in **butterfly subdivision**:
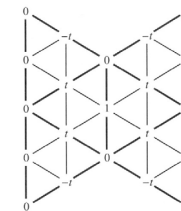
Setting $t=0$ gives the original polyhedron, and increasing small values of t makes the surface smoother, until $t=1/8$ when the surface is provably G^1.