Partial Differential Equations and Level-Set Methods in Image Sciences

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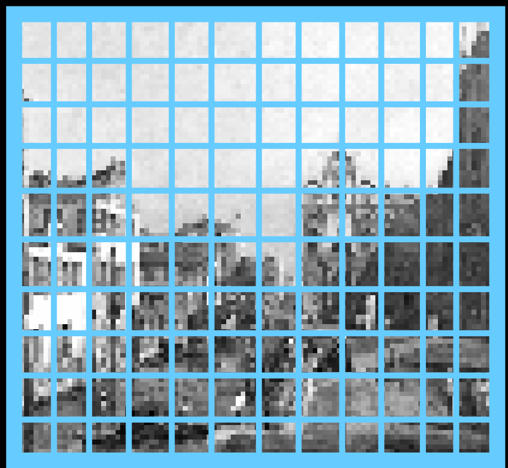
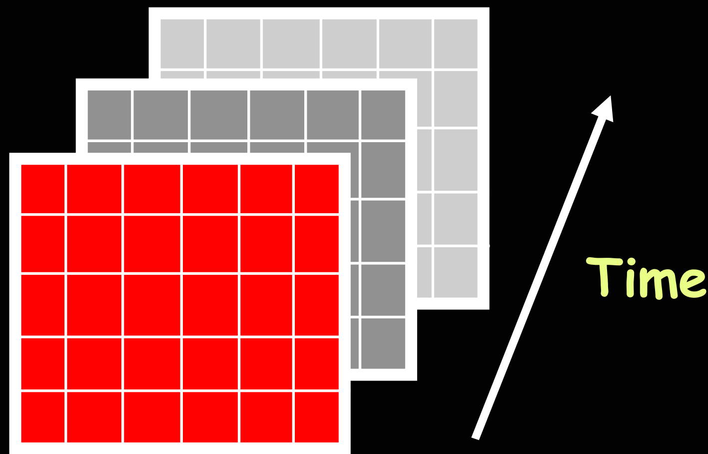
What is a discrete computer image?





Consequences of discrete image representations

- Classical image processing and computer vision is based on discrete mathematics (most of it)
  - Sums instead of integrals
  - Re-definition of classical continuous operators as a Laplacian, Minkowsky addition, etc
  - etc…

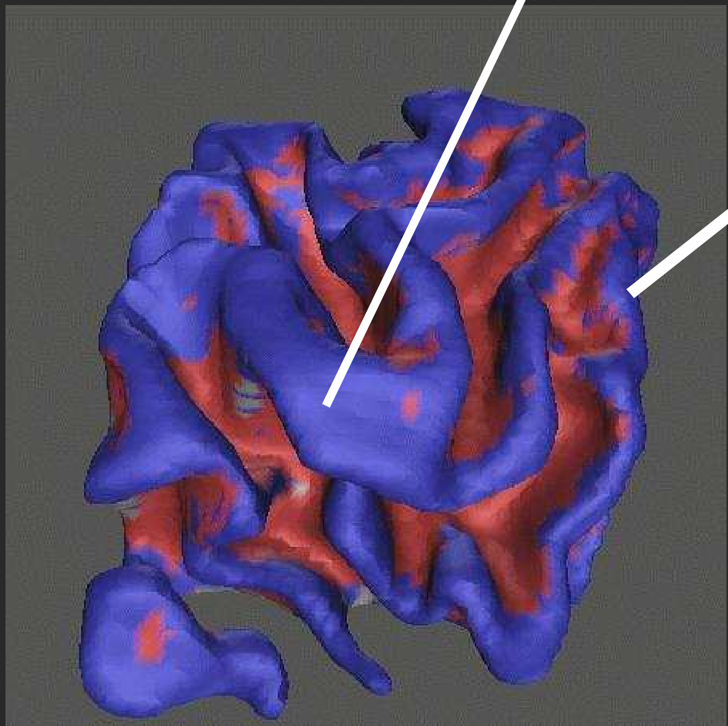
The PDE's approach

- Images are continuous objects
- Image processing is the results of iteration of infinitesimal operations: PDE's
- Differential geometry on images
- Computer image processing is based on numerical analysis

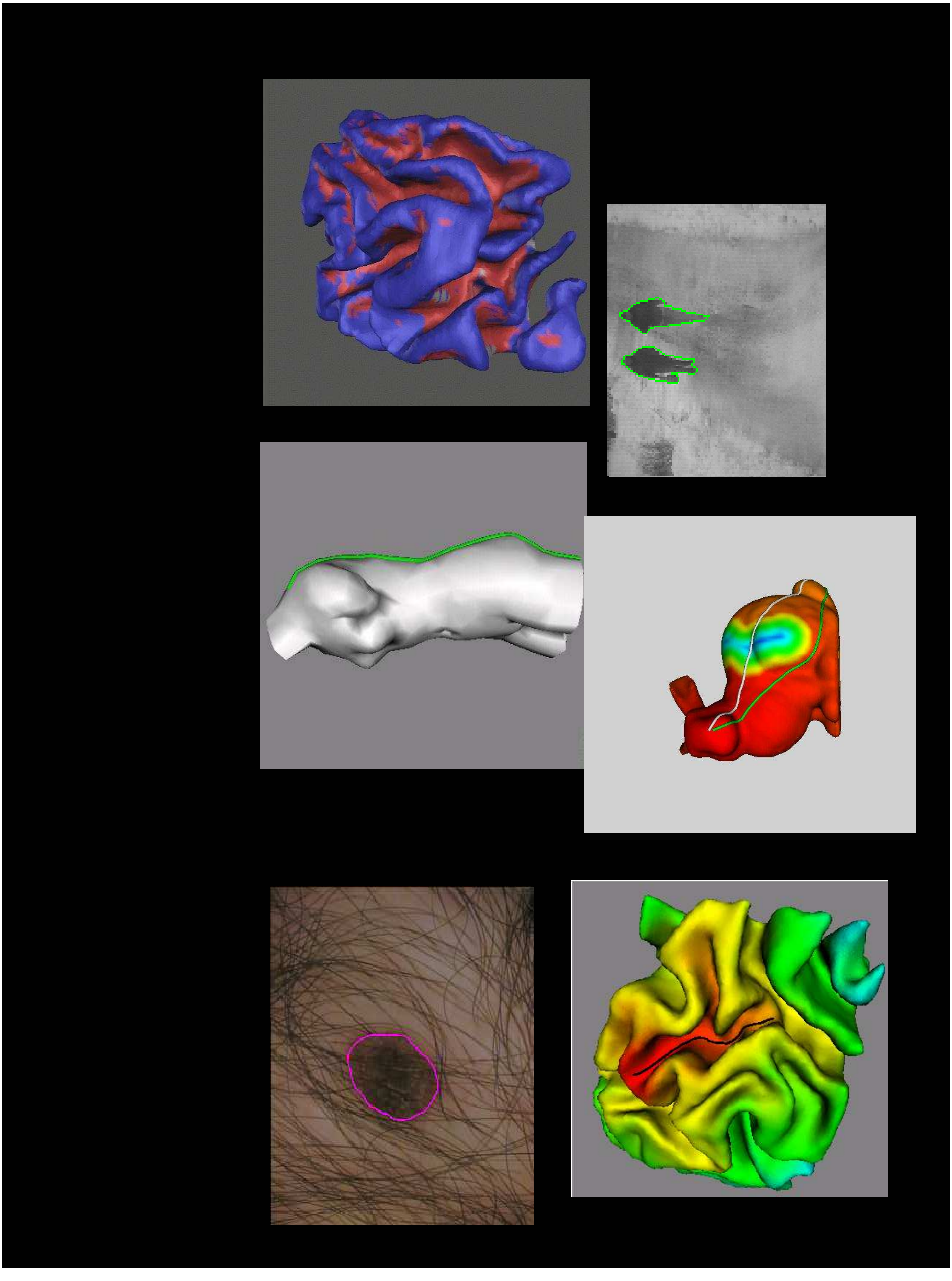
Why? Why Now? Who?

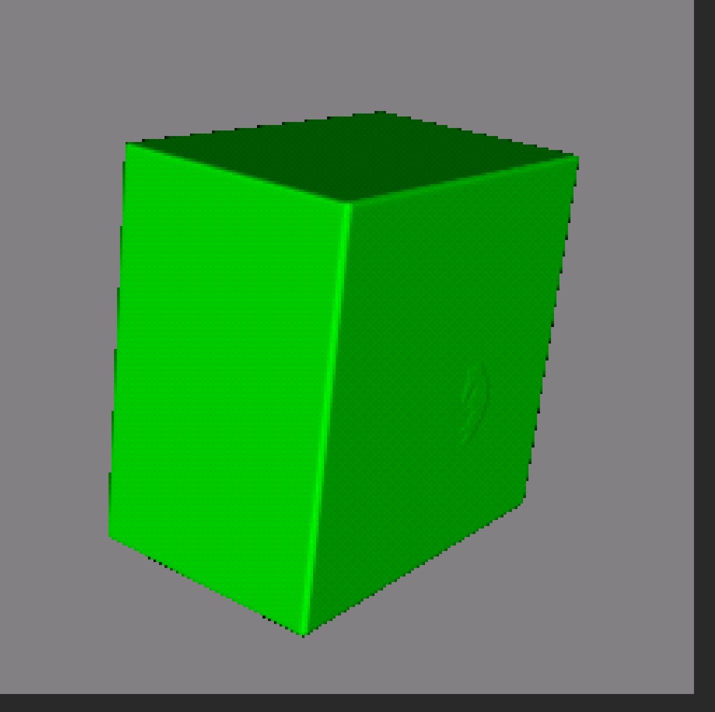
- **Why now:**
  - Computers!!!
  - People
- **Why:**
  - New concepts
  - Accuracy
  - Formal analysis (existence, uniqueness, etc)
- **Consequences:**
  - Many state of the art results

What is it?



V=F(curvatures, etc)



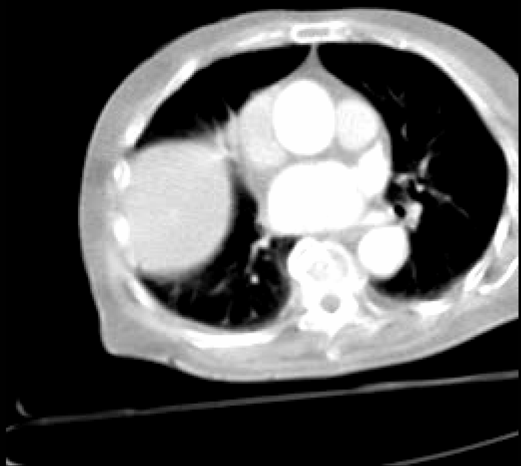


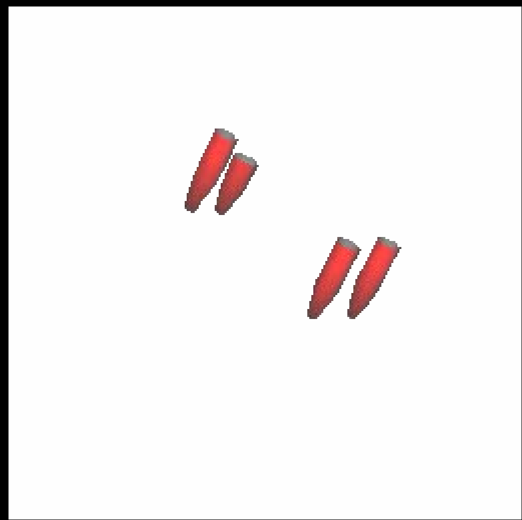
Heart segmentation from MRI data

(Malladi et al.)

Click each figure for a movie

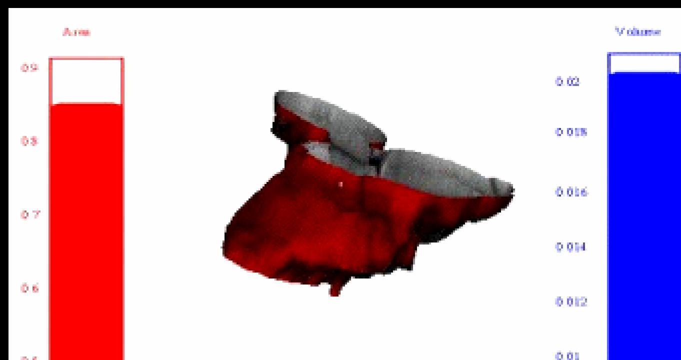
The data





Reconstruction

Reconstructed heart beating



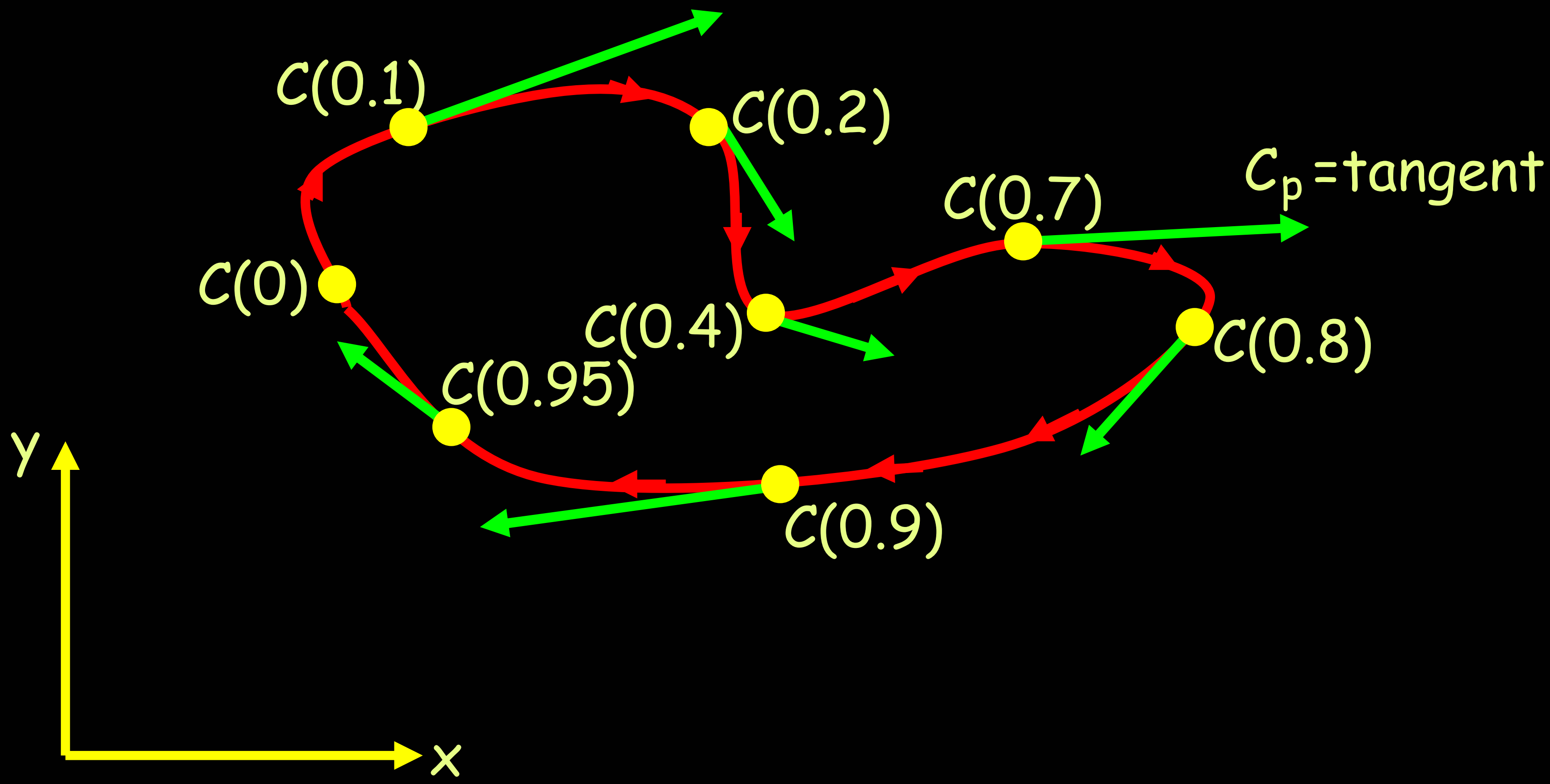
Heart and measurements

Introduction to Differential Geometry

Follows in part notes by R. Kimmel

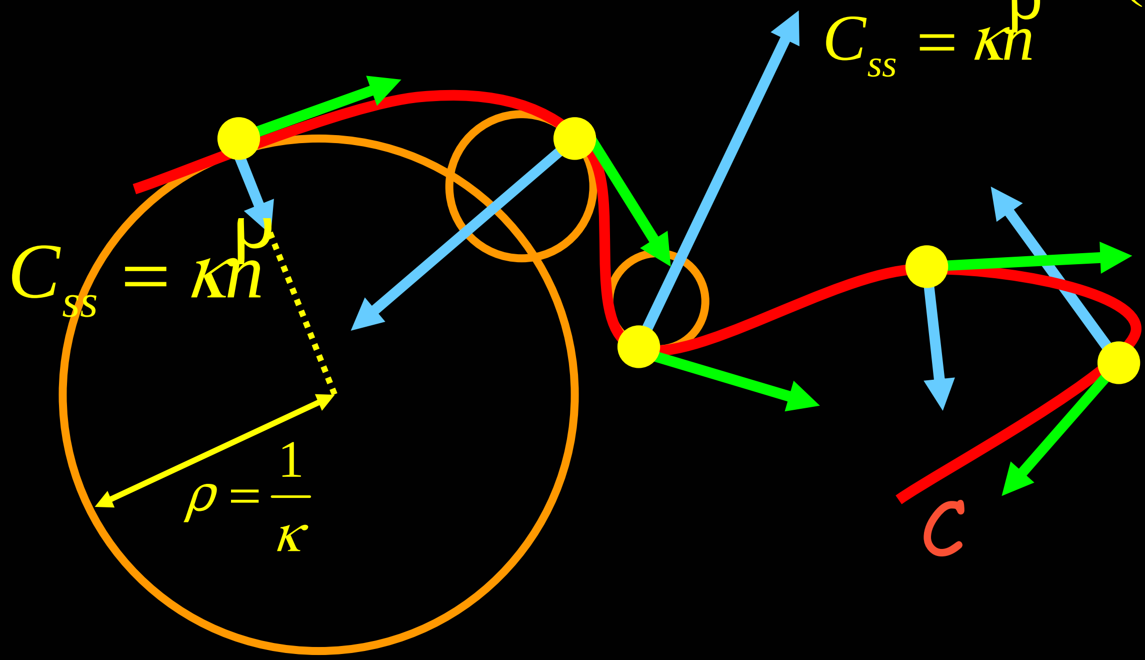
Planar Curves

- $C(p)=\{x(p),y(p)\}, \quad p \in [0,1]$



Arc-length and Curvature

$$s(p)=\int_0^p |C_p|\,dp \;\Rightarrow\; |C_s|=1, \qquad \left(\vec{t}=C_s=\frac{C_p}{|C_p|}\right)$$



Surface

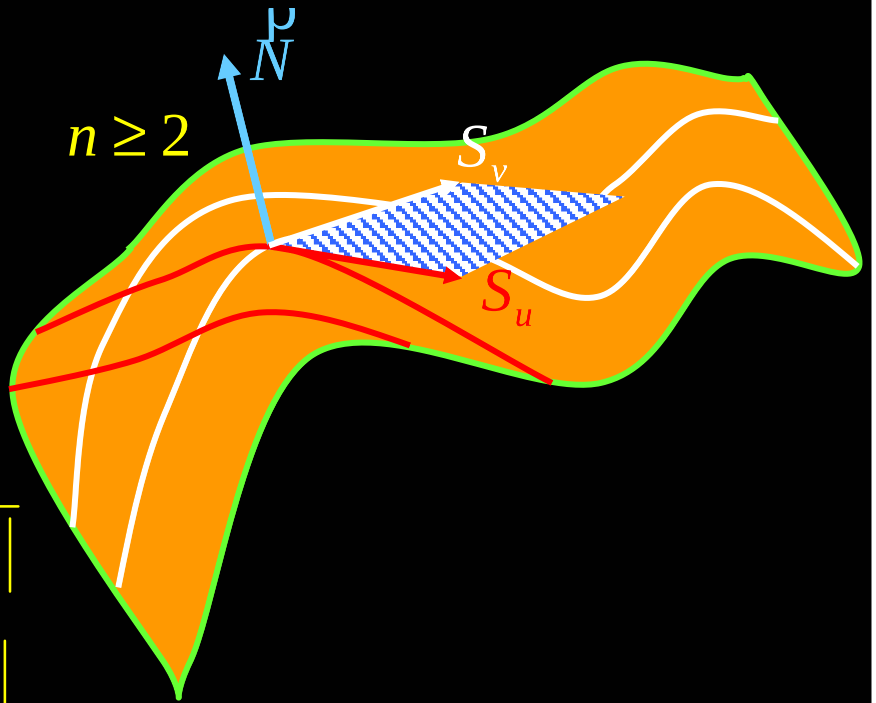
- A surface, $S:\Omega \subset \mathbf{R}^2 \to \mathrm{M}^n \quad n \geq 2$
- For example, in 3D

  $$S(u,v) = \{x(u,v), y(u,v), z(u,v)\}$$

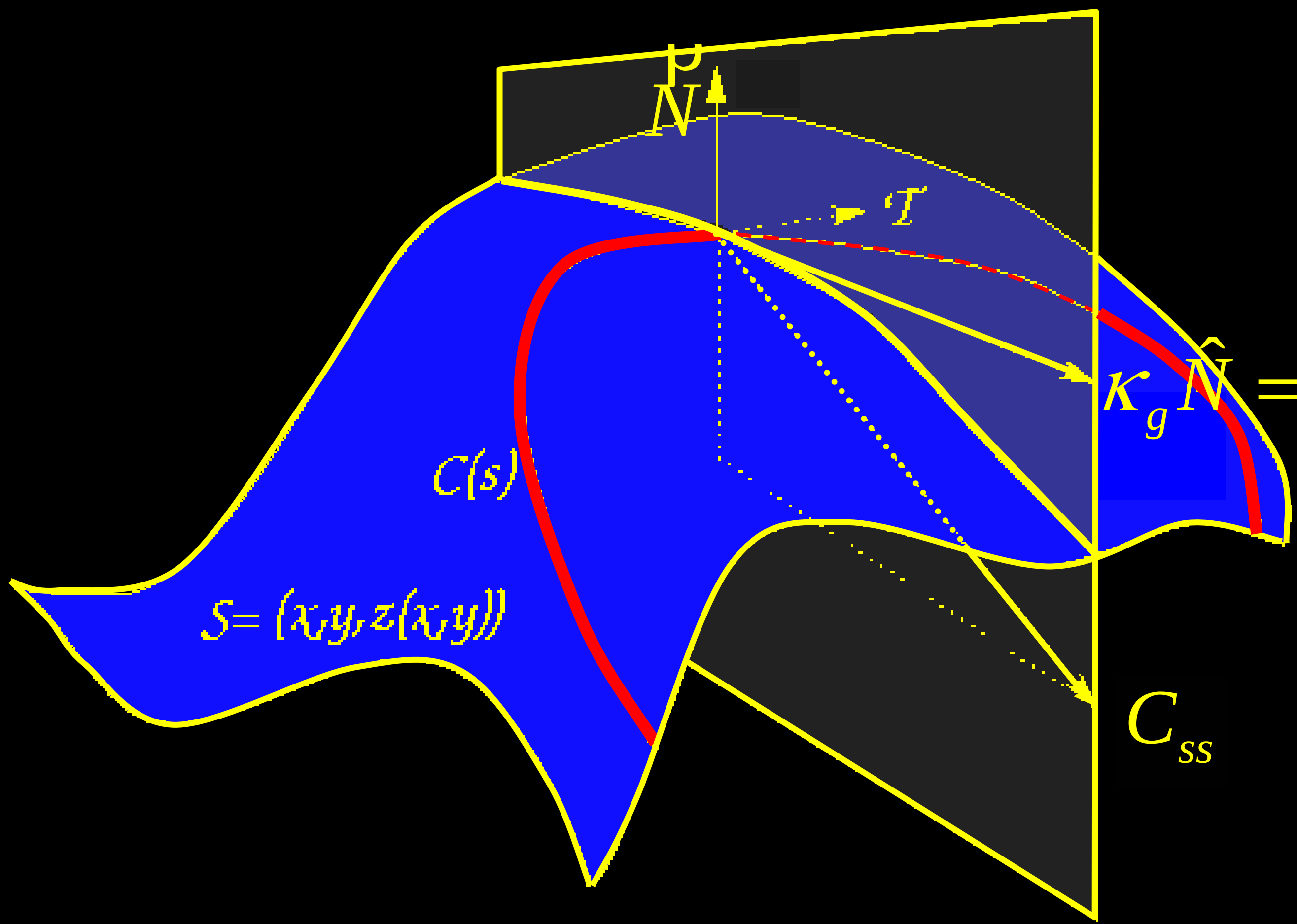
- Normal

  $$\vec{N} = \frac{S_u \times S_v}{|S_u \times S_v|}$$

- Area element $\quad dA = |S_u \times S_v|$
- Total area $\quad A = \iint |S_u \times S_v| \, du\, dv$



Curves on Surfaces: The Geodesic Curvature

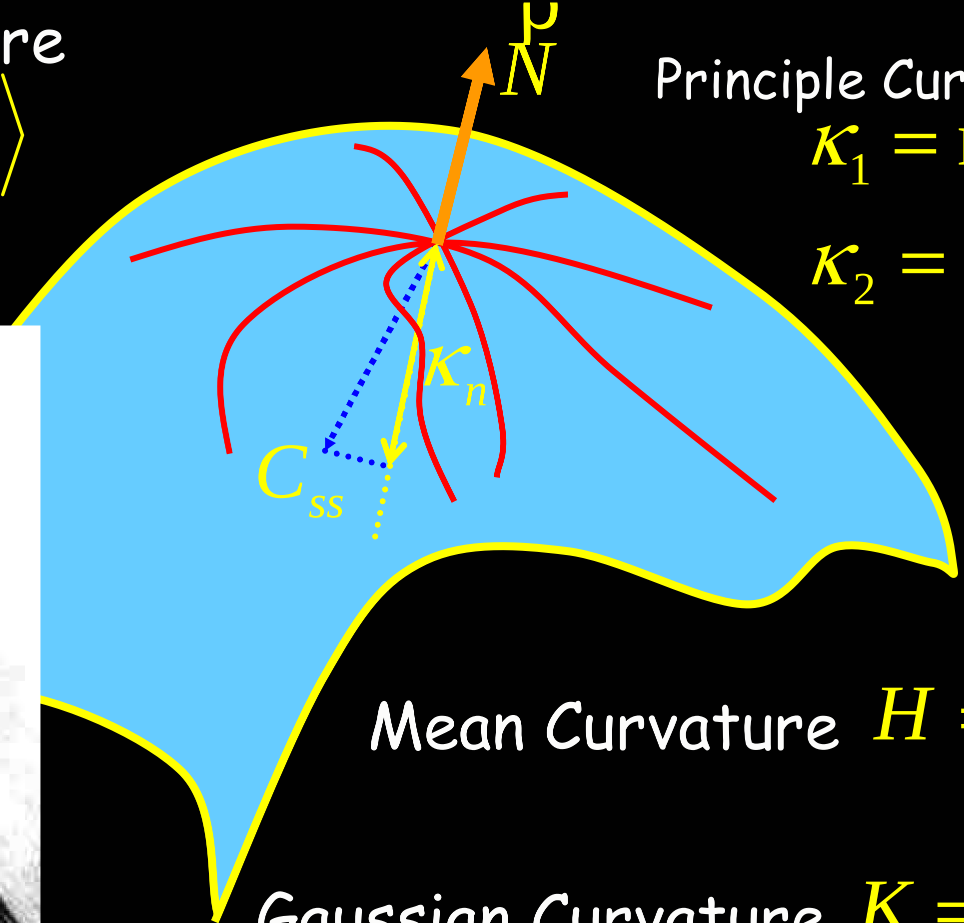


$$\kappa_g \hat{N} = C_{ss} - \left\langle C_{ss}, \vec{N} \right\rangle \vec{N}$$

Curves on Surfaces: The Geodesic Curvature

Normal Curvature

$$\kappa_n = \langle C_{ss}, \vec{N} \rangle$$



Principle Curvatures

$$\kappa_1 = \max_\theta(\kappa)$$

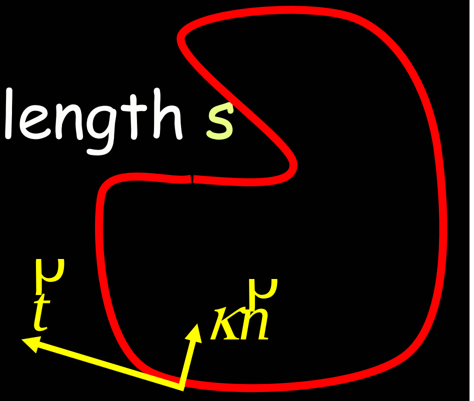
$$\kappa_2 = \min_\theta(\kappa)$$

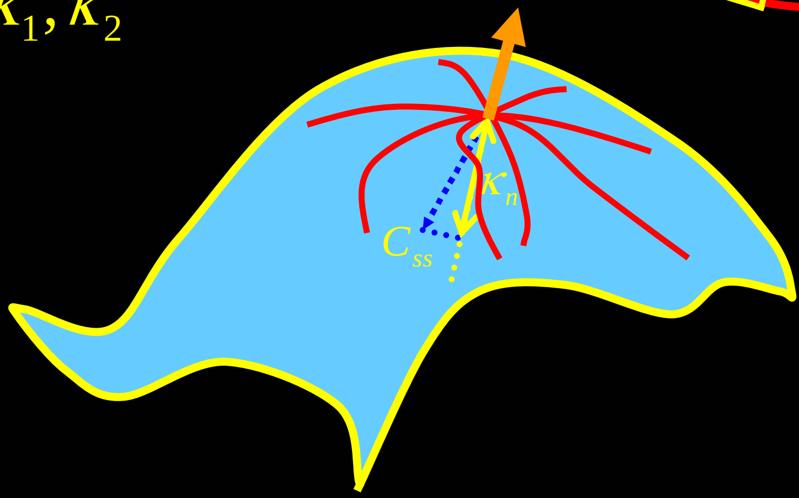
Mean Curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

Gaussian Curvature $K = \kappa_1 \kappa_2$

Geometric measures

- Curvature κ, normal $\vec{n}$, tangent $\vec{t}$, arc-length s
- Mean curvature H
- Gaussian curvature K
- principle curvatures κ_1, κ_2
- geodesic curvature κ_g
- normal curvature κ_n
- tangent plane T_p



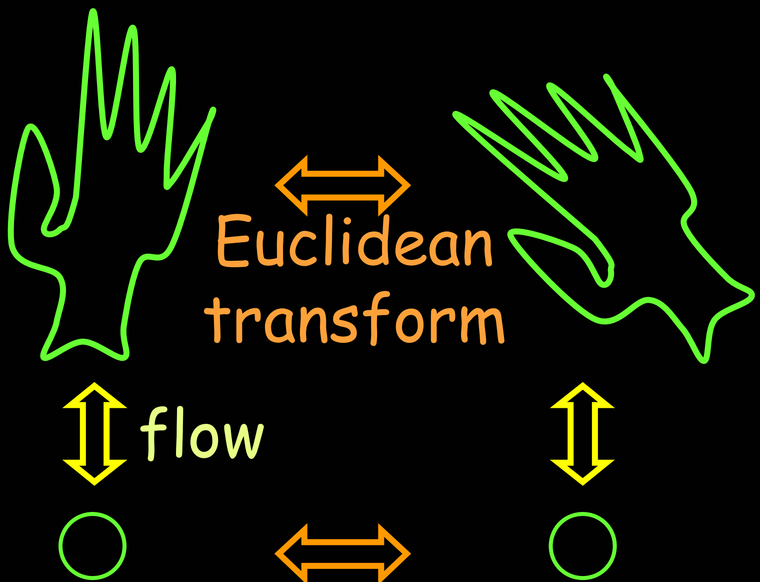


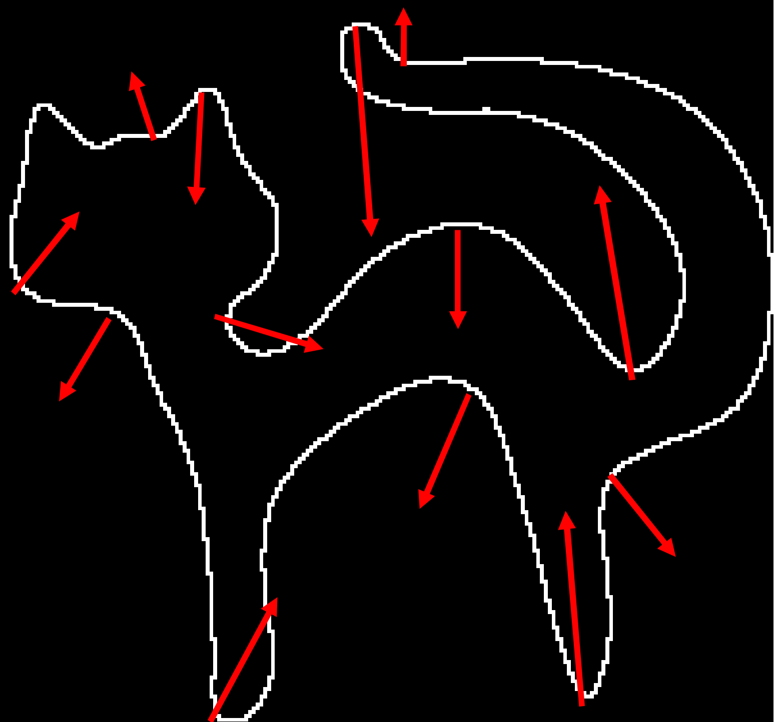
Planar Curve Evolution

Curvature flow $C_t = \kappa \vec{n}$

- Euclidean geometric heat equation

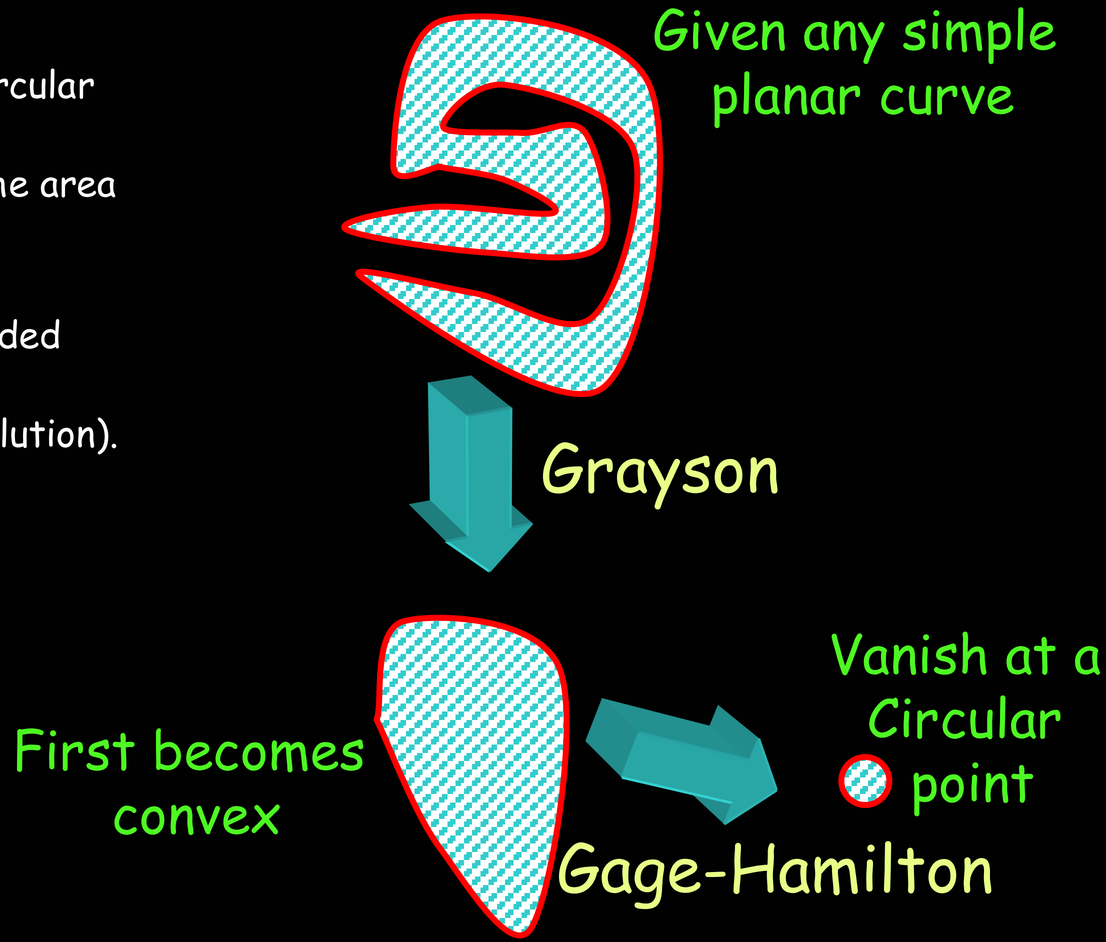
$$C_t = C_{ss} \quad \text{where } C_{ss} = \kappa \vec{n}$$

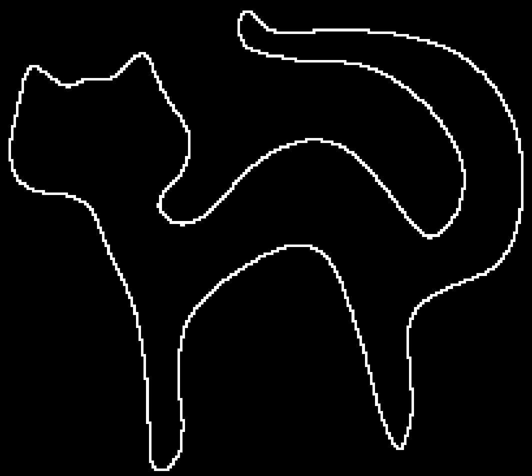




Curvature flow $C_t = \kappa \vec{n}$

- Takes any simple curve into a circular point
  in finite time proportional to the area inside
  the curve
- Embedding is preserved (embedded curves
  keep their order along the evolution).

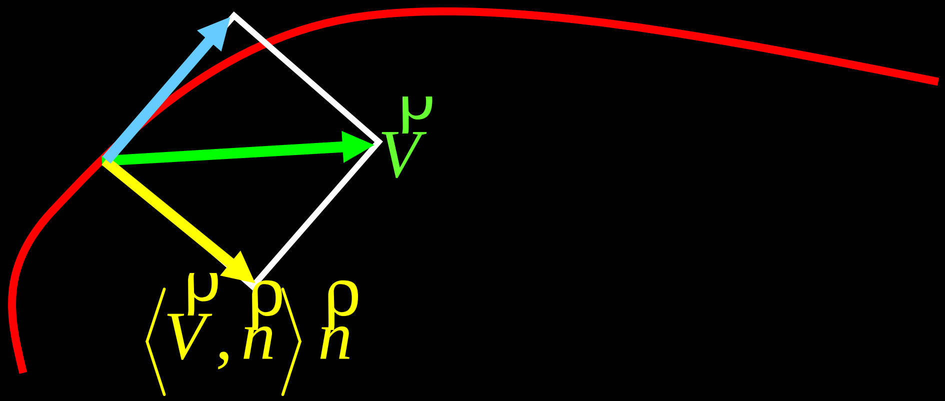




Important property

- Tangential components do not affect the geometry of an evolving curve

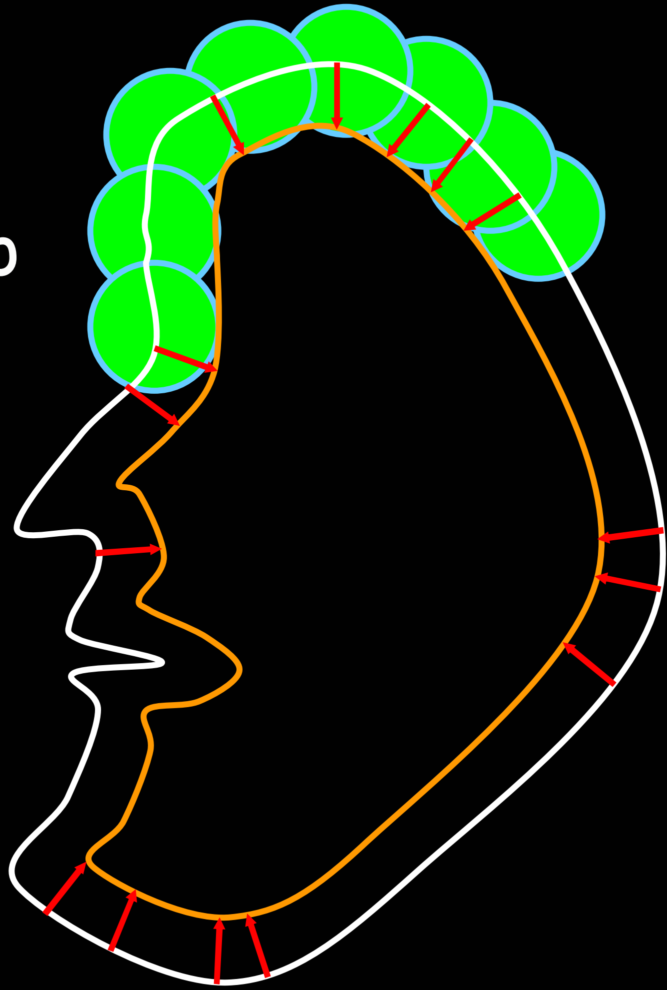
$$C_t = \vec{V} \Leftrightarrow C_t = \langle \vec{V}, \vec{n} \rangle \vec{n}$$



Constant flow

$$C_t = \vec{n}$$

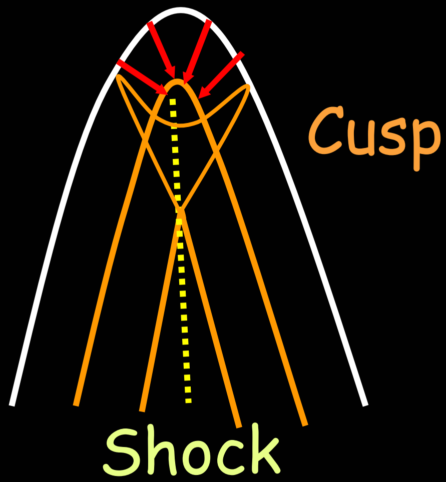
- Offset curves
- Level sets of distance map
- Equal-height contours of the distance transform
- Envelope of all disks of equal radius centered along the curve (Huygens principle)



Constant flow $C_t = \vec{n}$

- Offset curves

Change in topology



Introduction to
Calculus of Variations

Calculus of Variations

Generalization of Calculus that seeks to find the path, curve, surface, etc., for which a given Functional has a minimum or maximum.

Goal: find extrema values of integrals of the form

$$\int F(u, u_x)\,dx$$

It has an extremum only if the Euler-Lagrange Differential Equation is satisfied,

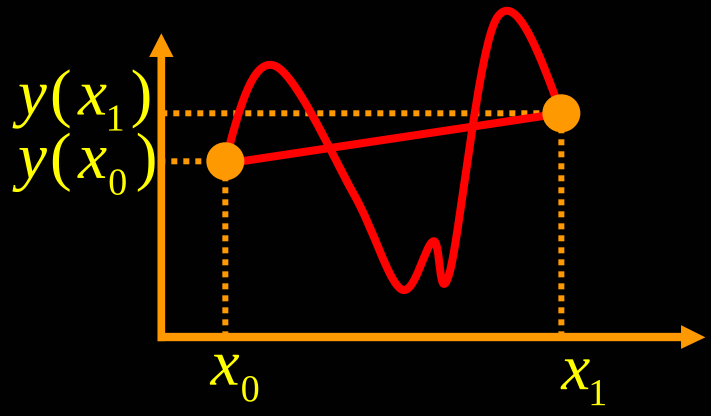
$$\left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right) F(u, u_x) = 0$$

Calculus of Variations

Example: Find the shape of the curve {x,y(x)} with shortest length:

$$\int_{x_0}^{x_1} \sqrt{1 + y_x^{\,2}}\, dx,$$

given *y(x0),y(x1)*



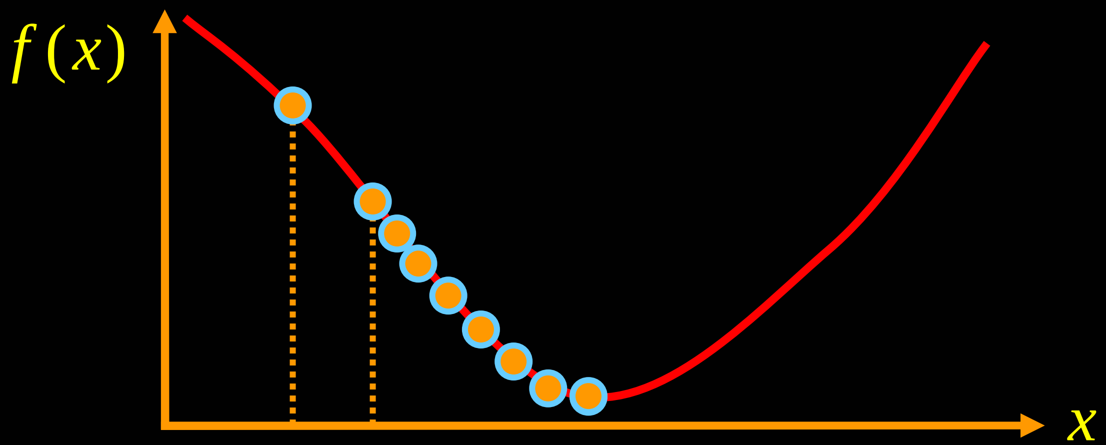
Solution: a differential equation that *y(x)* must satisfy,

$$\frac{y_{xx}}{\left(1 + y_x^{\,2}\right)^{3/2}} = 0 \quad \Rightarrow \quad y_x = a \;\Rightarrow\; y(x) = ax + b$$

Extrema points in calculus

$$\forall \eta : \lim_{\varepsilon \to 0} \left(\frac{df(x+\varepsilon\eta)}{d\varepsilon} \right) = 0 \Leftrightarrow \forall \eta : f_x(x)\eta = 0 \Leftrightarrow f_x(x) = 0$$

Gradient descent process $\quad x_t = -f_x$



$$x_0 \quad x_1 = x_0 - dt f_x(x_0)$$

Calculus of variations

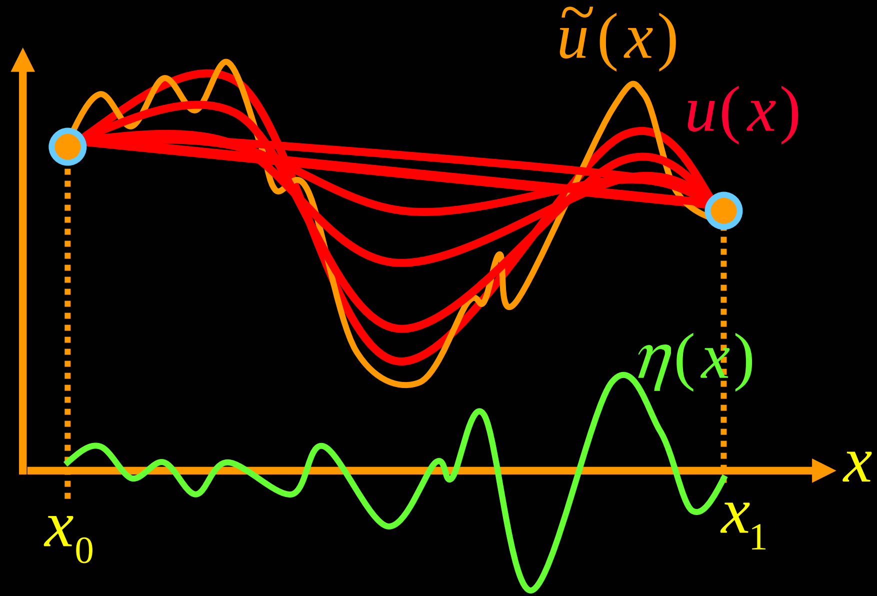
$$E(u(x)) = \int F(u, u_x)\,dx$$

$$\tilde{u}(x) = u(x) + \varepsilon\eta(x)$$

$$\forall\,\eta(x):\ \lim_{\varepsilon\to 0}\left(\frac{d}{d\varepsilon}\int F(\tilde{u},\tilde{u}_x)\,dx\right) \stackrel{?}{=} 0$$

$$\Downarrow$$

$$\frac{\delta E(u)}{\delta u} = \left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right)F(u, u_x)$$



Gradient descent process $u_t = -\dfrac{\delta E(u)}{\delta u}$

Euler Lagrange Equation

Proof. for fixed *u(x0),u(x1)* :

$$\int \frac{d}{d\varepsilon} F(\tilde{u}, \tilde{u}_x) dx = \int (F_{\tilde{u}} \tilde{u}_\varepsilon + F_{\tilde{u}_x} \tilde{u}_{x\varepsilon}) dx = \int (F_{\tilde{u}} \eta + F_{\tilde{u}_x} \eta_x) dx$$

$$= \int F_{\tilde{u}} \eta dx + F_{\tilde{u}_x} \eta \Big|_{x_0}^{x_1} - \int \eta \frac{d}{dx} (F_{\tilde{u}_x}) dx$$

$$= \int \left(F_{\tilde{u}} - \frac{d}{dx} (F_{\tilde{u}_x}) \right) \eta dx$$

Thus the Euler Lagrange equation is

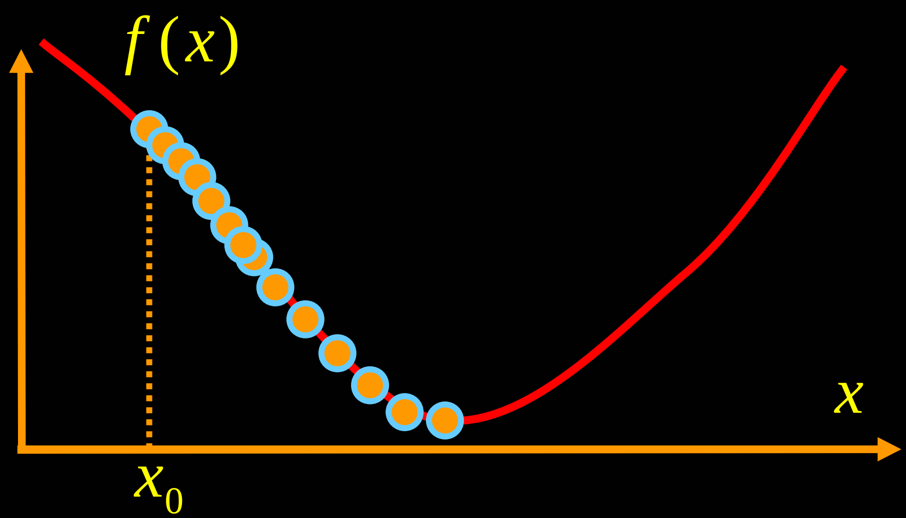
$$\left(\frac{\partial}{\partial u} - \frac{d}{dx} \frac{\partial}{\partial u_x} \right) F(u, u_x) = 0$$

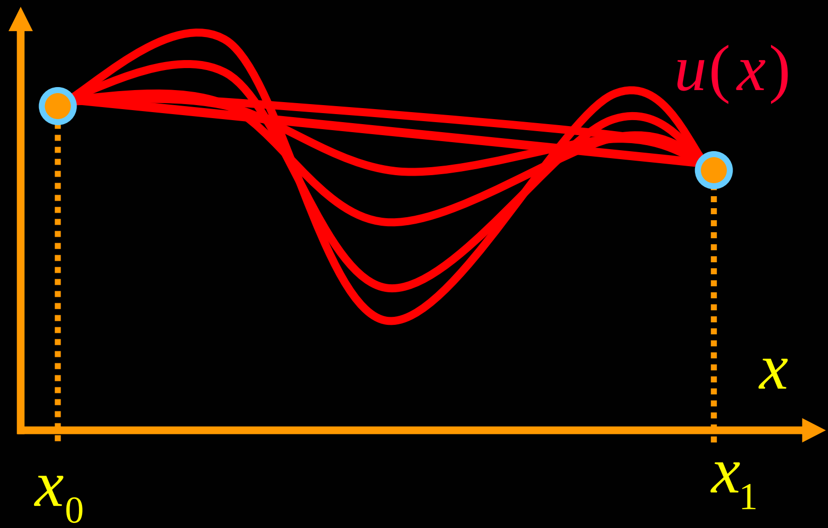
Conclusions

- Gradient descent process

Calculus $\quad \arg\min_{x} f(x) \Rightarrow x_t = -f_x$

Calculus of variations $\quad \arg\min_{u(x)} \underbrace{\int F(u,u_x)dx}_{E(u)} \Rightarrow u_t = -\underbrace{\frac{\delta E(u)}{\delta u}}_{\text{Euler-Lagrange}}$



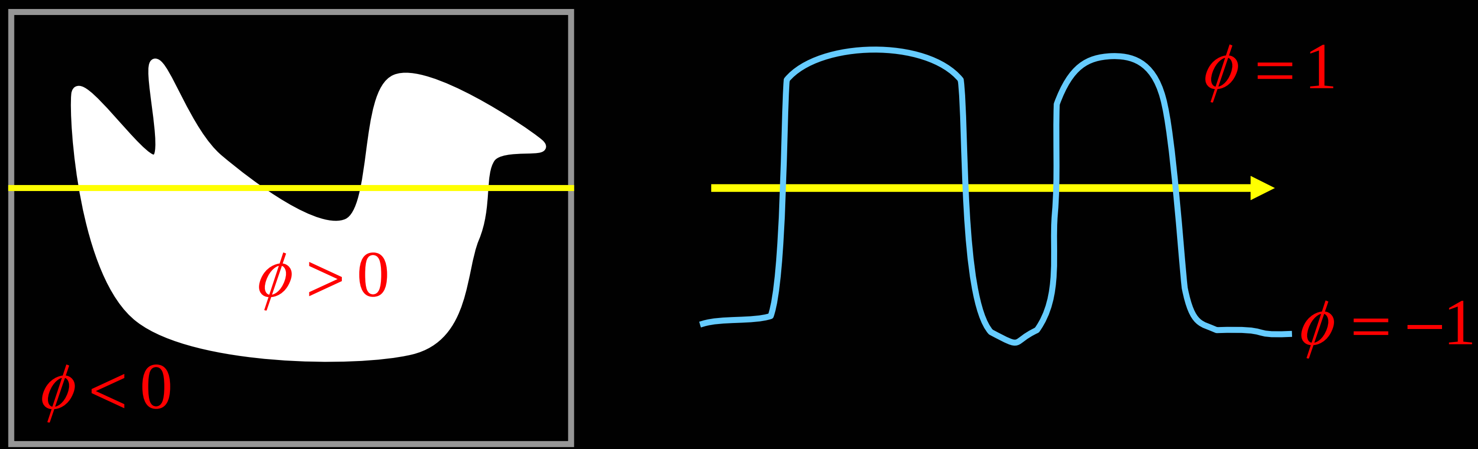


Level Set Formulation for Curve Evolution

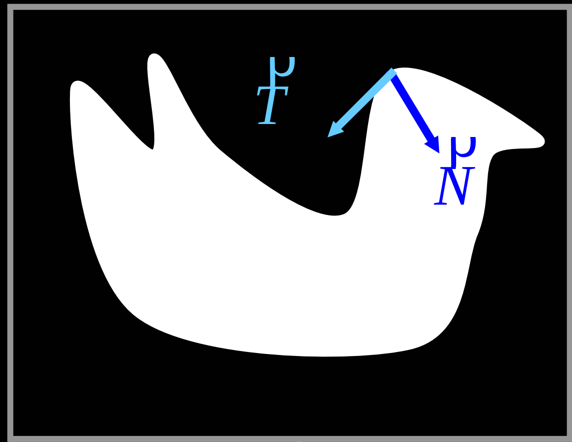
Implicit representation

Consider a closed planar curve $C(p): \mathbf{S}^1 \rightarrow \mathbf{R}^2$

The geometric trace of the curve can be alternatively represented implicitly as $C = \{(x, y) \mid \phi(x, y) = 0\}$



Properties of level sets



The level set normal

$$\vec{N} = -\frac{\nabla\phi}{|\nabla\phi|} \qquad \left(\vec{T} = \frac{\overline{\nabla}\phi}{|\nabla\phi|}\right)$$

Proof. Along the level sets we have zero change, that is $\phi_s = 0$, but by the chain rule

$$\phi_s(x,y) = \phi_x x_s + \phi_y y_s = \left\langle \nabla\phi, \vec{T} \right\rangle$$

So,

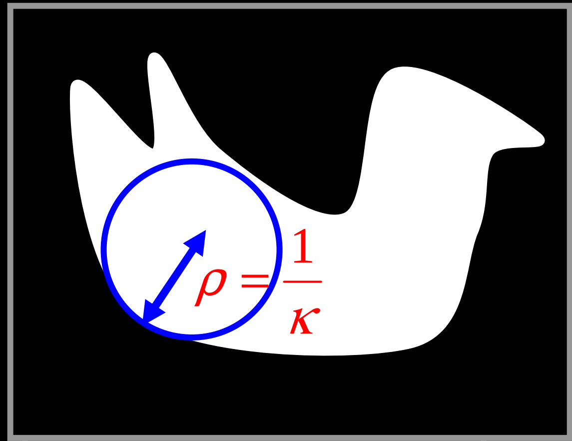
$$\left\langle \frac{\nabla\phi}{|\nabla\phi|}, \vec{T} \right\rangle = 0 \Rightarrow \frac{\nabla\phi}{|\nabla\phi|} \perp \vec{T} \Rightarrow \vec{N} = -\frac{\nabla\phi}{|\nabla\phi|}$$

∎

Properties of level sets

The level set curvature

$$\kappa = \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$



Proof. zero change along the level sets, $\phi_{ss} = 0$, also

$$\phi_{ss}(x,y) = \frac{d}{ds}(\phi_x x_s + \phi_y y_s) = \frac{d}{ds}\left\langle \nabla\phi, \vec{T} \right\rangle = \left\langle \frac{d}{ds}\nabla\phi, \vec{T} \right\rangle + \left\langle \nabla\phi, \kappa\vec{N} \right\rangle$$

So,

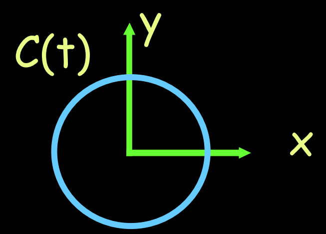
$$\kappa\left\langle \nabla\phi, \frac{\nabla\phi}{|\nabla\phi|} \right\rangle = \kappa|\nabla\phi| = -\left\langle [\phi_{xx}x_s + \phi_{xy}y_s, \phi_{xy}x_s + \phi_{yy}y_s], \frac{\overline{\nabla}\phi}{|\nabla\phi|} \right\rangle$$

$$= -\left\langle \left[\left\langle \nabla\phi_x, \vec{T} \right\rangle, \left\langle \nabla\phi_y, \vec{T} \right\rangle\right], \frac{\overline{\nabla}\phi}{|\nabla\phi|} \right\rangle = -\left\langle \left[\left\langle \nabla\phi_x, \frac{\overline{\nabla}\phi}{|\nabla\phi|} \right\rangle, \left\langle \nabla\phi_y, \frac{\overline{\nabla}\phi}{|\nabla\phi|} \right\rangle\right], \frac{\overline{\nabla}\phi}{|\nabla\phi|} \right\rangle \ldots$$

■

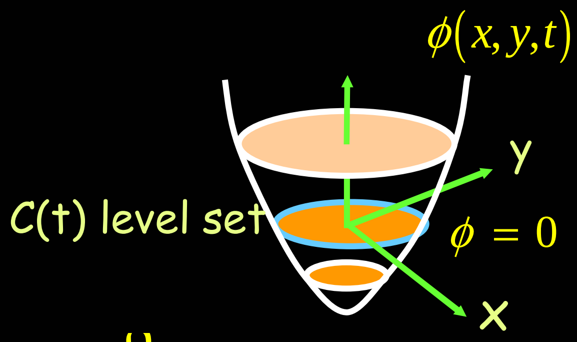
Level Set Formulation

(Osher-Sethian)



$C = \{(x, y) : \phi(x, y) = 0\}$ implicit representation of *C*

Then, $$\frac{dC}{dt} = V\vec{N} \quad \Leftrightarrow \quad \frac{d\phi}{dt} = V\,|\nabla\phi|$$



Proof. By the chain rule

$$0 = \frac{\partial\phi(x, y; t)}{\partial t} = \phi_x x_t + \phi_y y_t + \phi_t$$

Then,

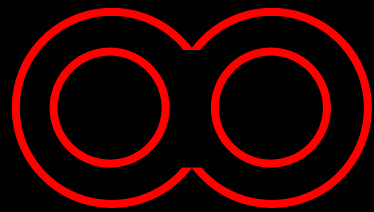
$$-\phi_t = \phi_x x_t + \phi_y y_t = \langle \nabla\phi, C_t \rangle = \langle \nabla\phi, V\vec{N} \rangle = V\langle \nabla\phi, \vec{N} \rangle$$

Recall that $\vec{N} = -\dfrac{\nabla\phi}{|\nabla\phi|}$, and $-V\langle \nabla\phi, \vec{N} \rangle = V\left\langle \nabla\phi, \dfrac{\nabla\phi}{|\nabla\phi|} \right\rangle = V\,|\nabla\phi|$

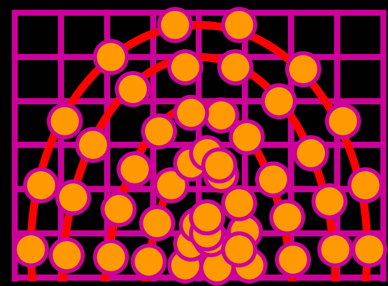
$\Rightarrow$ $$\boxed{\phi_t = V\,|\nabla\phi|}$$

Level Set Formulation

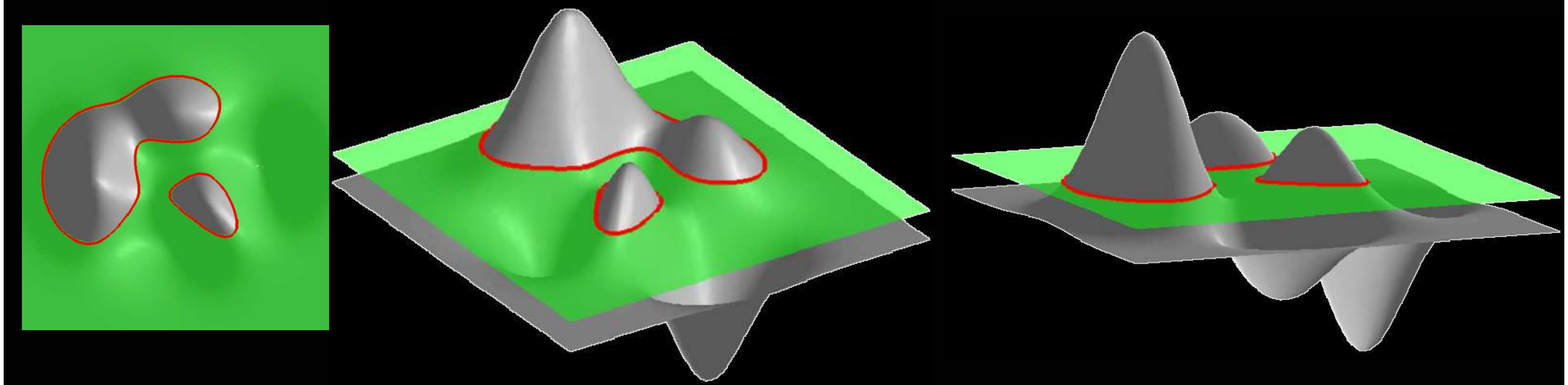
- Handles changes in topology



- Numeric grid points never collide or drift apart.



- Natural philosophy for dealing with gray level images.



Numerical Considerations

- Finite difference approximation.
- Order of approximation, truncation error, stencil.
- (Differential) conservation laws.
- Entropy condition and vanishing viscosity.
- Consistent, monotone, upwind scheme.
- CFL condition (stability examples)

Numerical Considerations

Central derivative

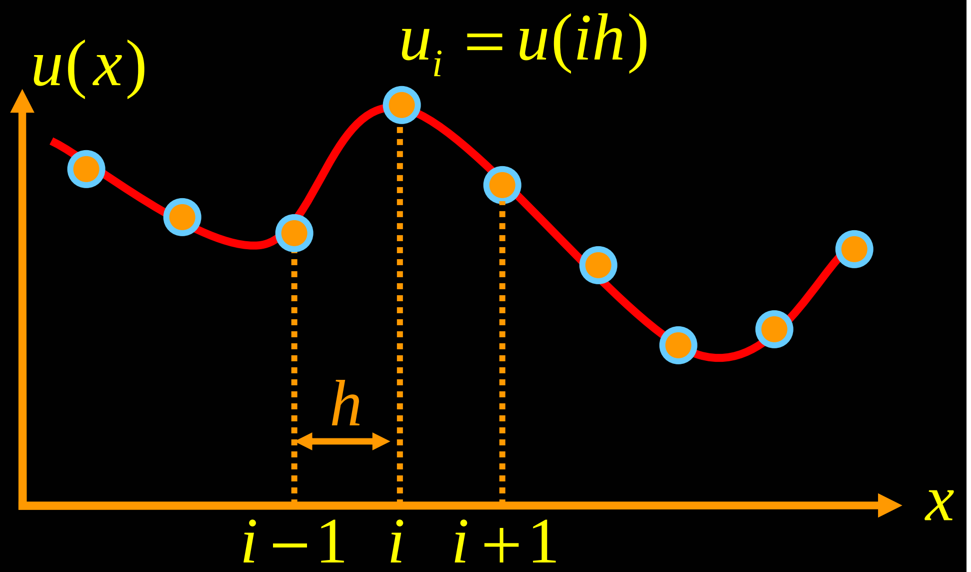
$$D_x u \equiv \frac{u_{i+1} - u_{i-1}}{2h}$$

Forward derivative

$$D_x^+ u \equiv \frac{u_{i+1} - u_i}{h}$$

Backward derivative

$$D_x^- u \equiv \frac{u_i - u_{i-1}}{h}$$



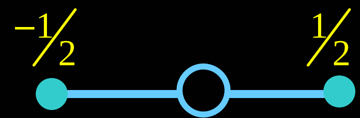
Truncation Error

Taylor expansion about $x=ih$

$$u_{i+1} = u(ih+h) = u(ih) + hu'(ih) + \tfrac{1}{2!}h^2u''(ih) + O(h^3)$$

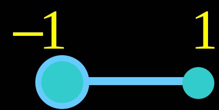
$$u_{i-1} = u(ih-h) = u(ih) - hu'(ih) + \tfrac{1}{2!}h^2u''(ih) + O(h^3)$$

$$D_x u_i = u'(ih) + O(h^2)$$

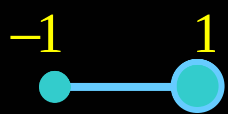


Stencils

$$D_x^+ u_i = u'(ih) + O(h)$$

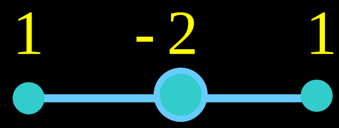


$$D_x^- u_i = u'(ih) + O(h)$$

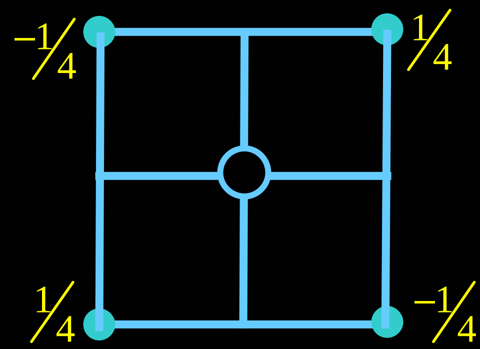


Numerical Approximations

$$D_{xx}u \equiv \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$



$$D_{xy}u \equiv \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4h^2}$$

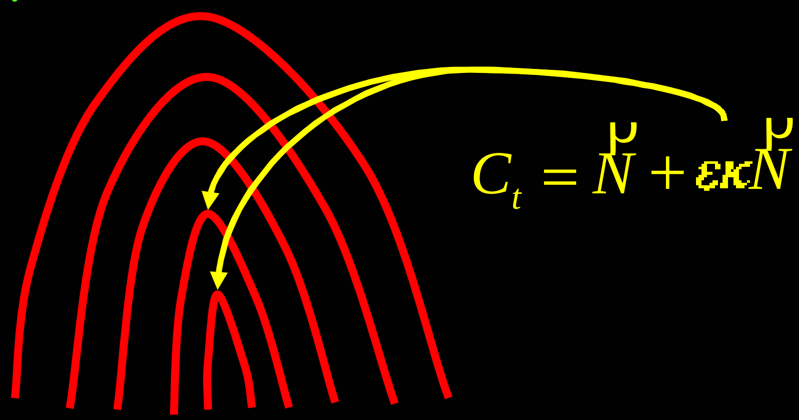


Hamilton-Jacobi

In 1D: HJ=Hyperbolic conservation laws
In 2D: just the `flavor'…

Vanishing viscosity, $\lim_{\varepsilon \to 0}$ of $$u_t + \left(H(u)\right)_x = \varepsilon u_{xx}$$

The `entropy condition' selected the `weak solution' that is the `vanishing viscosity solution' also known as `entropy solution'.



General GBM Framework For Object Segmentation

GBM

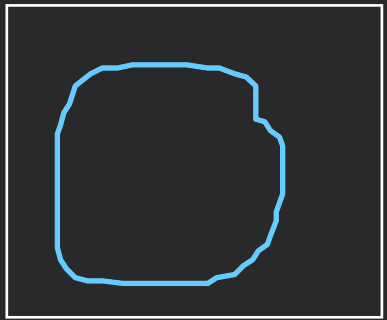
Introduction

- Goal: Object detection
- Approach: Curve/surface deformation
  - Geometry dependent regularization
  - Image dependent velocity

Notation

- Deforming curve:
- Image:

$$C(p):[0,1]\rightarrow R^2$$



$$I:[0,1]\times[0,1]\rightarrow R^2 \quad (R^N)$$

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Basic active contours approach

- Terzopoulos *et al.*, Cohen *et al.*

$$E(C) = \lambda \int |C(p)|^2 \, dp + \gamma \int |C'(p)|^2 \, dp - \int |\nabla I(C)| \, dp$$

Geodesic active contours
(Caselles-Kimmel-Sapiro, ICCV '95)

$$E(C) \;=\; \lambda \int |C'(p)|^2 \, dp \;+\; \gamma \int |C''(p)|^2 \, dp \;-\; \int |\nabla I(C)| \, dp$$

- Generalize image dependent energy
- Eliminate high order smoothness term
- Equal internal and external energies
- Maupertuis and Fermat principles of dynamical systems

$$E(C) \;=\; \int g[|\nabla I(C(s))|] \, ds$$

Kimmel-Caselles-Sapiro, Yezzi et al., Paragios-Deriche, etc

Geodesic computation

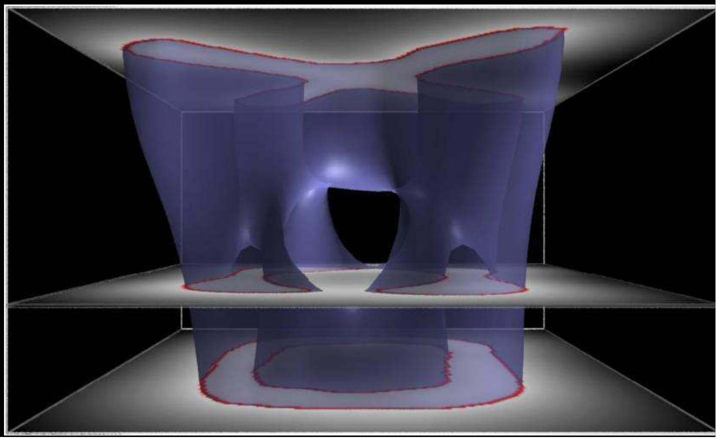
- **Gradient-descent**

$$E(C) = \int ds \quad \Rightarrow \quad \frac{\partial C}{\partial t} = \kappa \vec{N}$$

$$E(C) = \int g[|\nabla I(C(s))|]\, ds \quad \Rightarrow \quad \frac{\partial C}{\partial t} = g\,\kappa \vec{N} - \nabla g \cdot \vec{N}$$

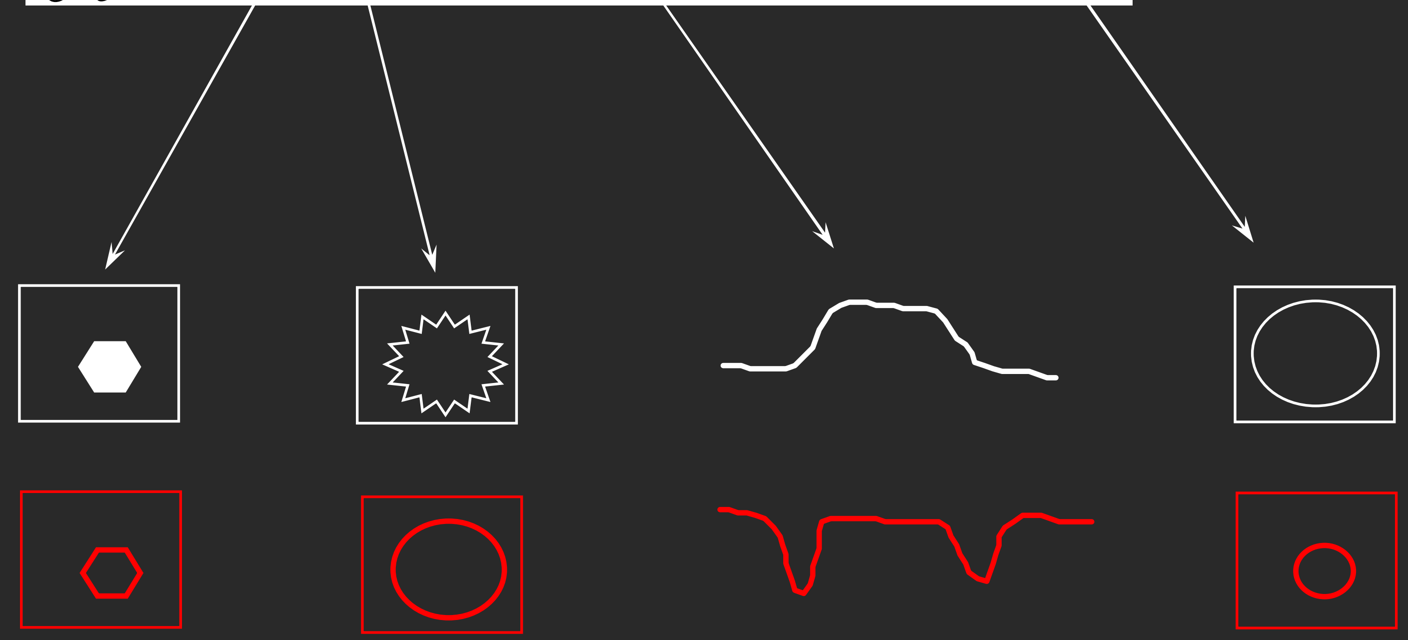
Follows Osher-Sethian:

$$\frac{\partial C}{\partial t} = \beta \vec{N} \quad \Rightarrow \quad \frac{\partial \Phi}{\partial t} = \beta |\nabla \Phi|$$



Further geometric interpretation

$$\frac{\partial C}{\partial t} = g\,\kappa\,\vec{N} - \nabla g \cdot \vec{N} \qquad (+\,g\,\vec{N}\,)$$



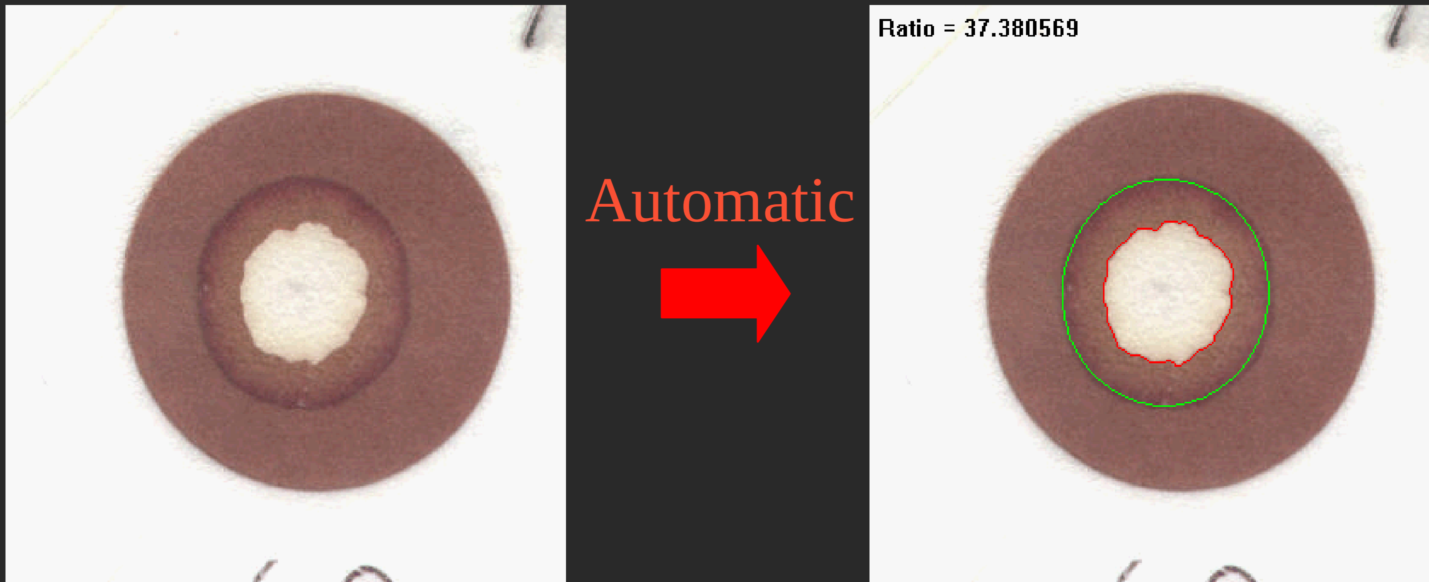
Model correctness

- **Theorem:** The deformation is independent of the level-sets embedding function
- **Theorem:** There is a unique solution to the flow in the viscosity framework
- **Theorem:** The curve converges to ideal objects when present in the image

Automatic skin lesion segmentation via GBM

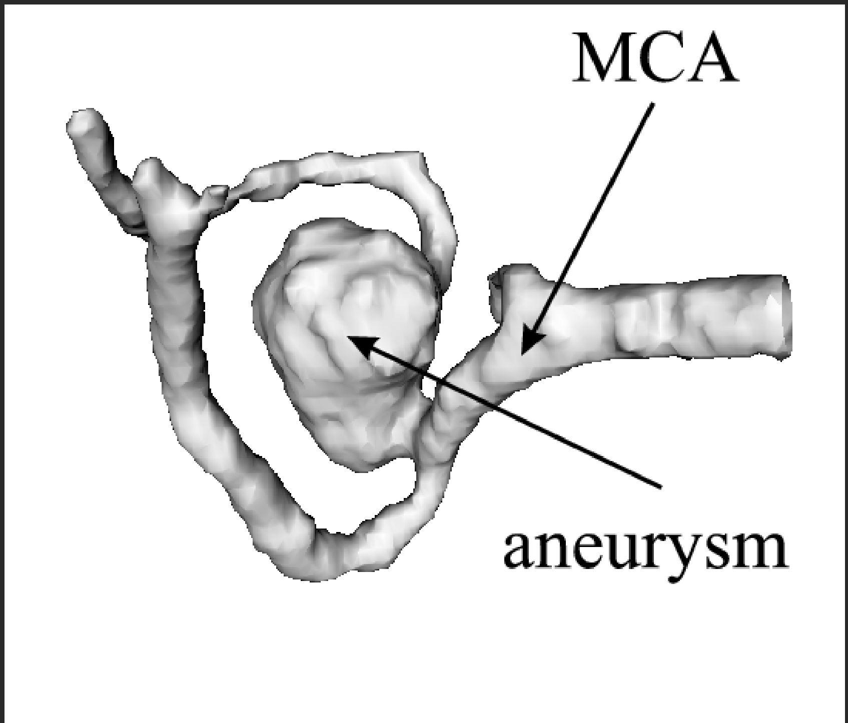
D. H. Chung and G. Sapiro (IEEE-TMI 2000)

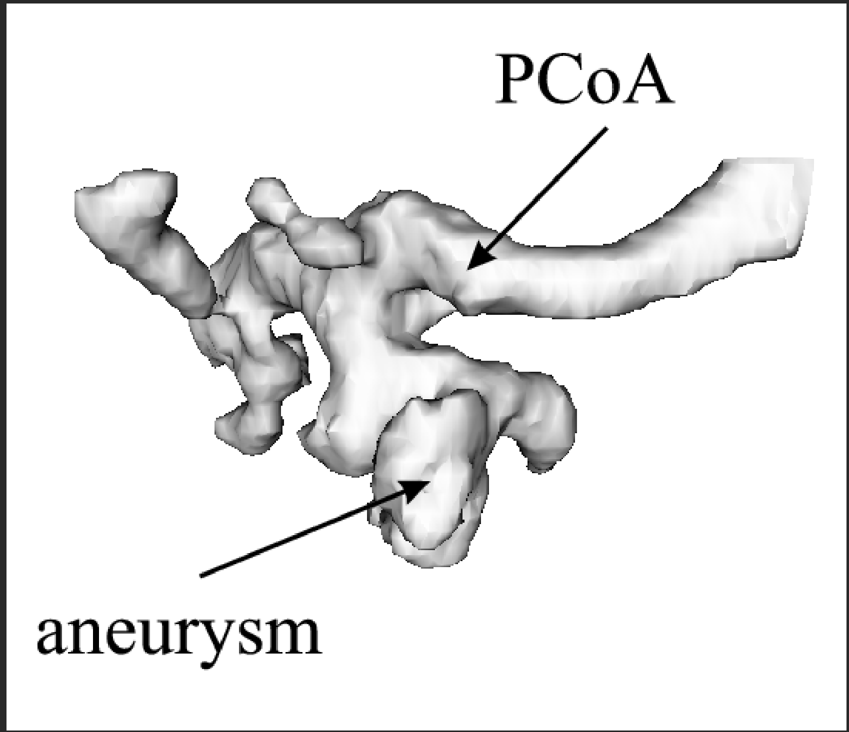
A non-invasive test to aid in the diagnosis of cystic fibrosis: Automatic chloride patch/sensor analysis



- Ratio between red and green areas is in correlation with chloride concentration, aiding in the diagnosis of CF
- Collaboration with local industry (PolyChrome Medical), and Medical School (Prof. Warren Warwick) , performed by Bartesaghi & Sapiro

Example





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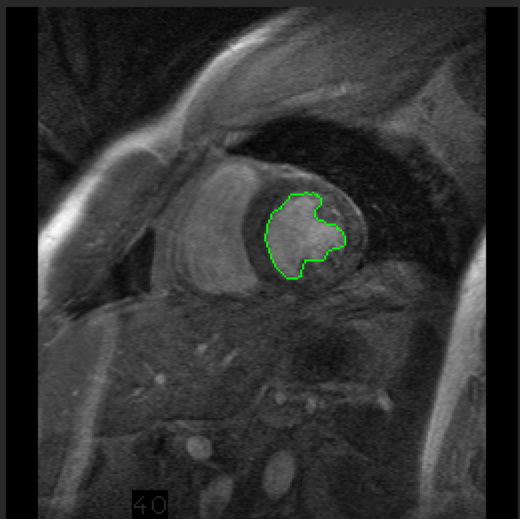
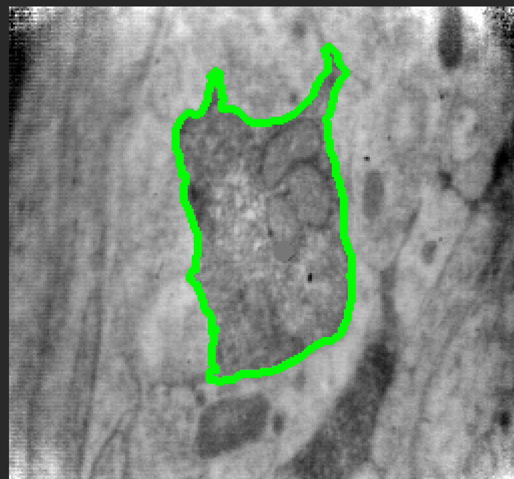
Morphing active contours





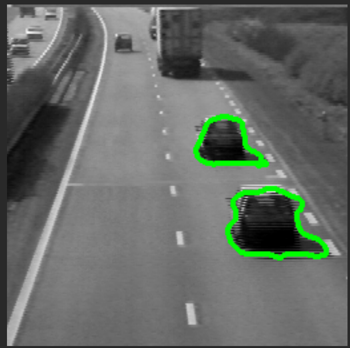
Bertalmio-Sapiro-Randall, IEEE-PAMI

Show it to me!!!

The main problem and our goal
(Bertalmio-Sapiro-Randall, IEEE-PAMI)

- Problem: Track objects (video or 3D slices)

Frame n



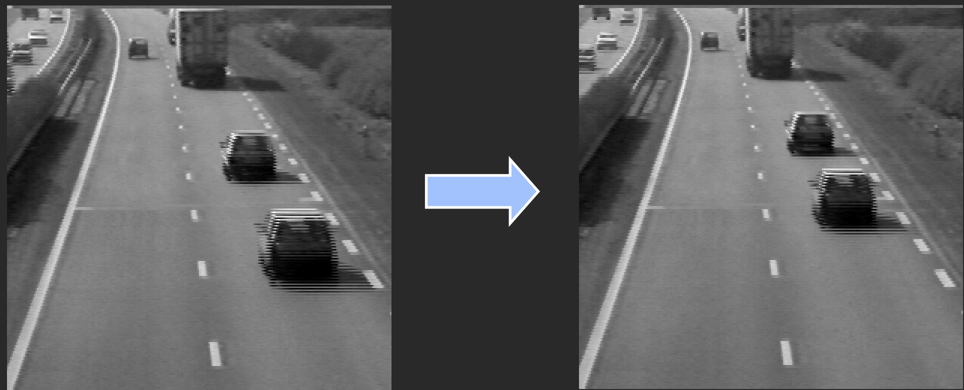
Frame n+1

- Our Goal:
  - Simple (no learning, or statistics, etc)
  - Handle objects merging and splitting (changes in the topology)
  - Accurate
  - Computationally efficient

- See also Paragios-Deriche, ICCV '97/'99, CVPR '99

Basic Idea

- Coupled Partial Differential Equations
  - Equation 1: Morphing Equation



  - Equation 2: Tracking Equation

The Equations

$$\frac{\partial F_n}{\partial t} = \underbrace{\Delta(F_n, F_{n+1})}_{\text{Normal velocity}} \left|\nabla F_n\right|$$

Morphing eq.

$$\frac{\partial U}{\partial t} = \Delta(F_n, F_{n+1}) \underbrace{(\vec{N}_{F_n} \bullet \vec{N}_U)}_{\text{Projection term}} \left|\nabla U\right|$$

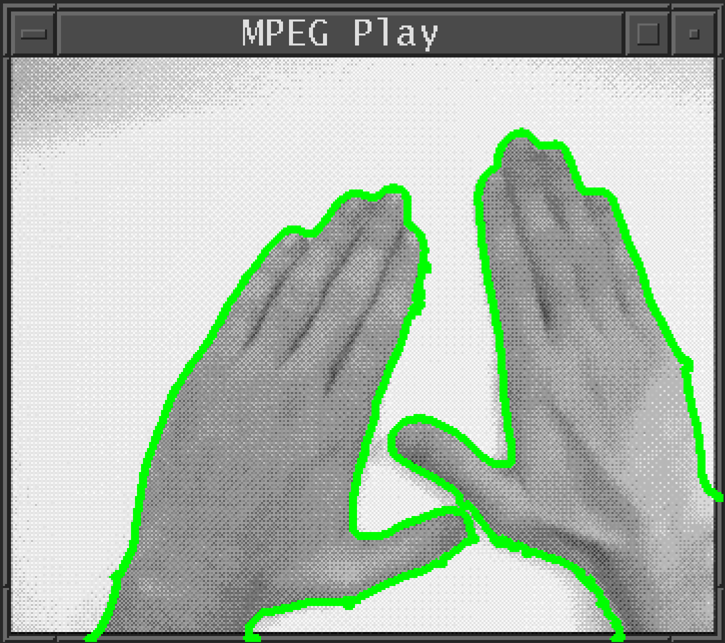
Tracking eq.

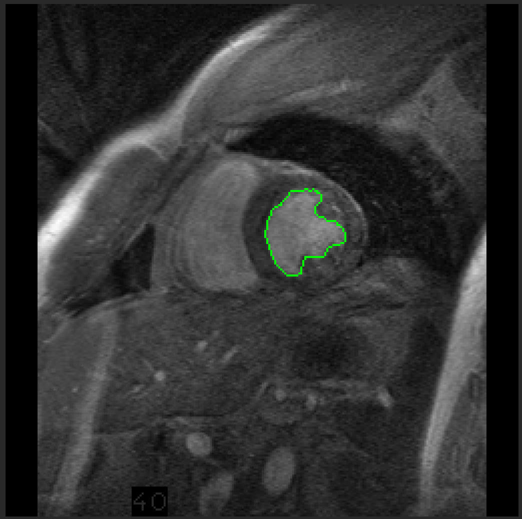
F_i : features in frame i (e.g., gray - value, edges)

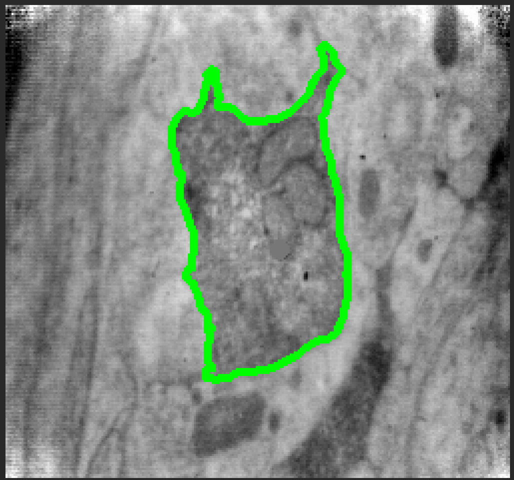
Δ : discrepancy function (e.g., absolute difference)

U : its zero - level set is the boundary to track.

$$\Delta \xrightarrow{t\to\infty} 0 \quad \Rightarrow \quad F_n \to F_{n+1},\ C_n \to C_{n+1}$$







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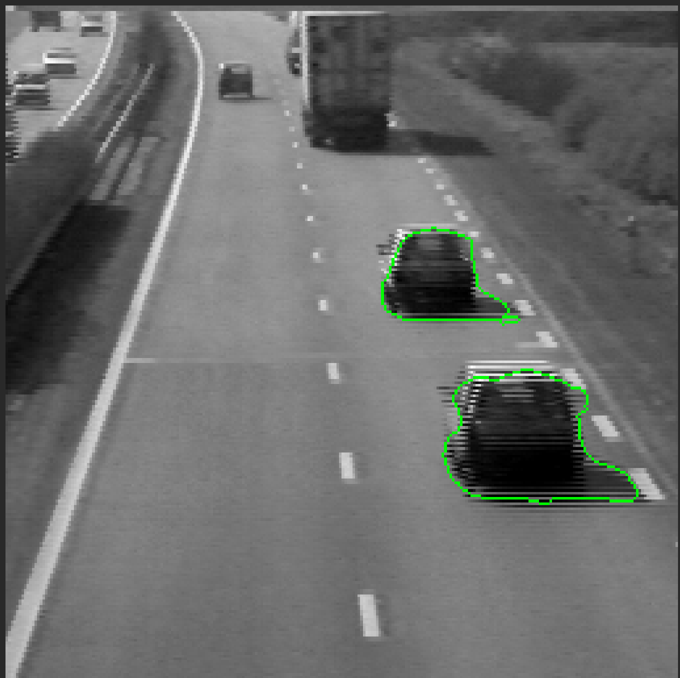
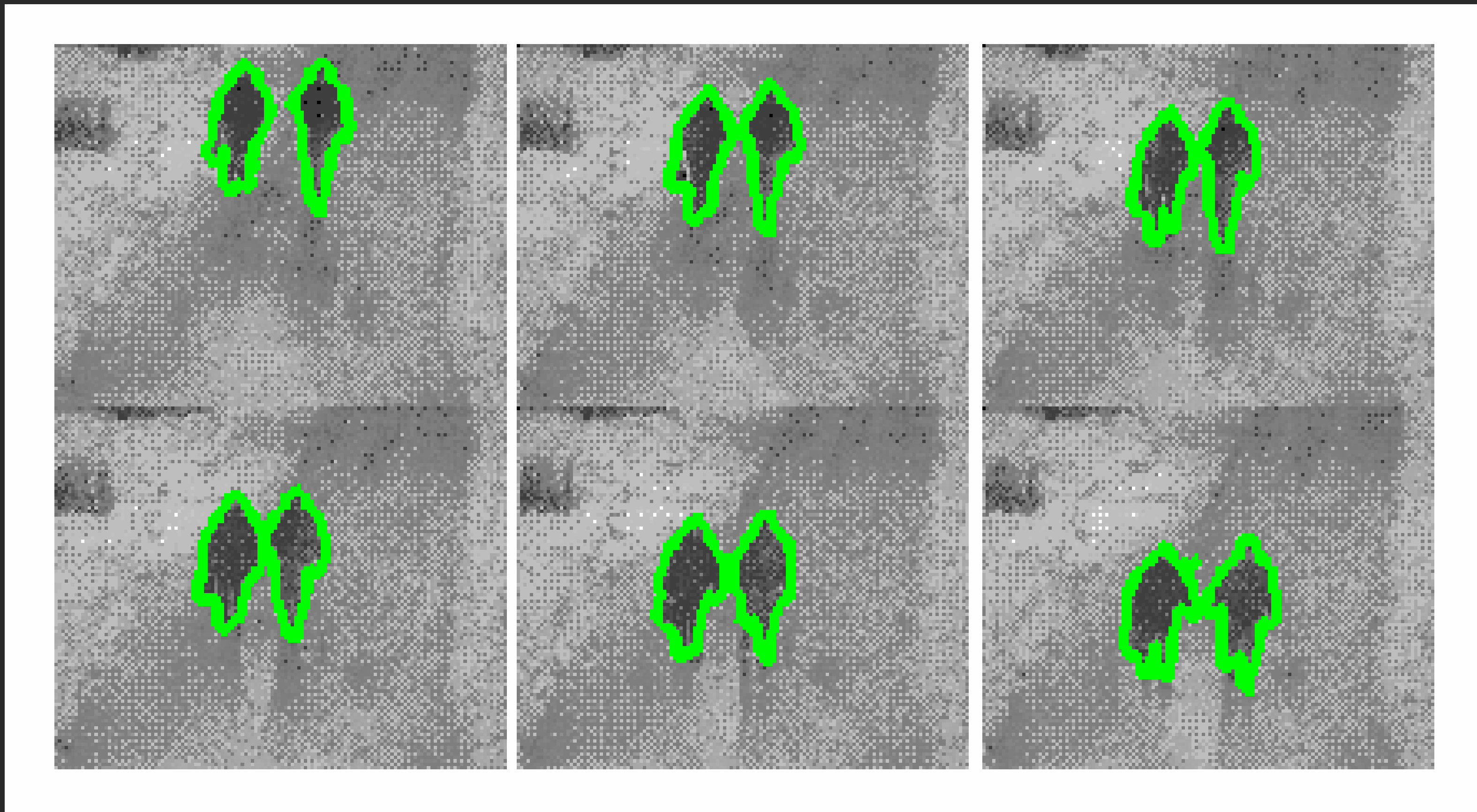
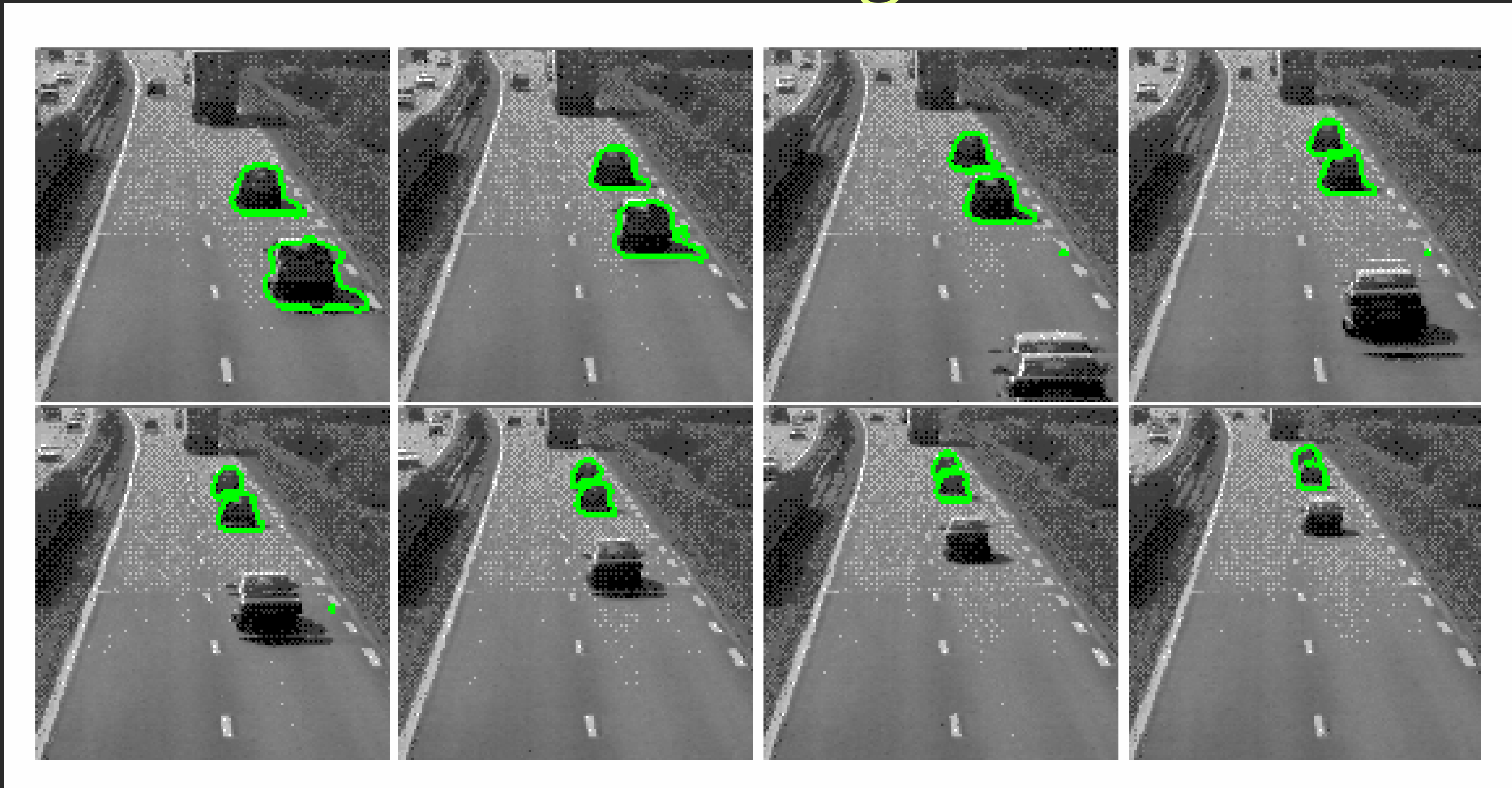


Tracking



Tracking



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