10/18/10

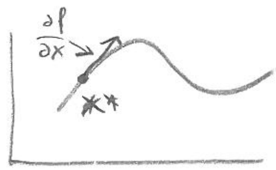
LINEARIZATION AND LOCAL STABILITY

$\dot{x} = f(x)$

Taylor series:

$$\dot{x} = f(x) = f(x^*) + \left.\frac{\partial f}{\partial x}\right|_{x=x^*} (x - x^*) + \text{higher order terms.}$$

1D:



For equilibrium point $f(x^*) = 0$

To first order:

$$\dot{x} \cong \left.\frac{\partial f}{\partial x}\right|_{x=x^*} (x - x^*) = \underbrace{\left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_n} & & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\right|_{x=x^*}}_{J} (x - x^*) = J(x - x^*)$$

Theorem: If x^* is an asymptotically stable equilibrium of $\dot{x} = J(x - x^*)$, then it is a locally asymptotically stable equilibrium of $\dot{x} = f(x)$.

Theorem: If the real part of the eigenvalues of J are strictly negative, then x^* is a stable equilibrium of $\dot{x} = J(x - x^*)$ and a locally stable equilibrium of $\dot{x} = f(x)$.

EXAMPLE:

$$\dot{x}_1 = -x_1 - x_2 + x_1 x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -x_2 - x_1 x_2 = f_2(x_1, x_2)$$

$$J = \left.\begin{pmatrix} -1 + x_2 & -1 + x_1 \\ -x_2 & -1 - x_1 \end{pmatrix}\right|_{x = x^*}$$

$$x^* = (0,0) \qquad x^* = (-1, 1/2)$$

$$J_{(0,0)} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$\det(J - \lambda I) = \det\begin{pmatrix} -1-\lambda & -1 \\ 0 & -1-\lambda \end{pmatrix} = (1+\lambda)^2 = 0 \quad \Rightarrow \lambda = -1$$

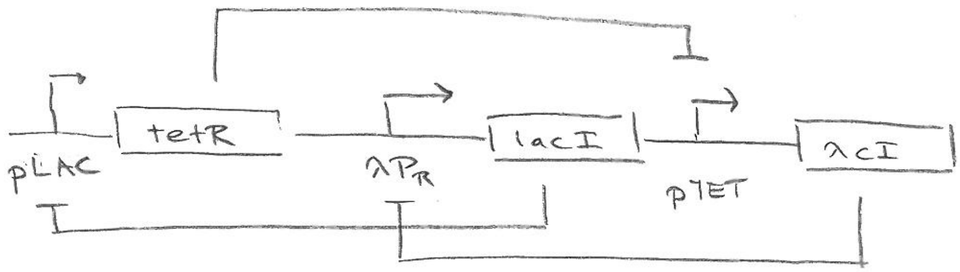
$\Rightarrow$ stable equilibrium

$$J_{(-1, 1/2)} = \begin{pmatrix} -1/2 & -2 \\ -1/2 & 0 \end{pmatrix} \quad \Rightarrow \lambda = (-1 \pm \sqrt{17})/2$$

$\Rightarrow$ equilibrium is not stable

4.3 A TRANSCRIPTIONAL OSCILLATOR

(ELOWITZ + LEIBLER, NATURE 403, 336 (2000))

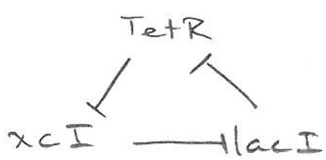


(PLASMID 1: OSCILLATOR)

(PLASMID 2: READOUT)

λcI, λP_R : REPRESSOR / PROMOTER FROM λ PHAGE

CORE CIRCUIT:



$$\dot{m}_i = -\gamma_m m_i + \frac{\alpha}{(1 + P_j^n/K)} + \alpha_0 \qquad i = \text{lacI}, \text{tetR}, \text{cI}$$

$$j = \text{cI}, \text{lacI}, \text{tetR}$$

$$\dot{P}_i = \alpha_p m_i - \gamma_p P_i$$

(NOTE: SLIGHTLY DIFFERENT NOTATION IN PAPER)

MAYBE:

* DISCUSS PHASE SPACE OF OSCILLATOR
* SHOW MOVIE (DOWNLOAD FROM ELOWITZ WEB PAGE)