Finite Model Theory – Homework 1

March 25, 2018

1 Zero-One Law

1. (0 points)

   (a) The Zero-One Law for First Order Logic *fails* if the vocabulary contains constants. For example, if we have one binary relation E (describing a graph) and a constant a then:

   $$\mu_n(E(a,a)) = \frac{1}{2}$$

   Where did we use the assumption that there are no constants in the proof of the zero-one law?

   (b) Consider the following two sentences:

   $$\varphi = \forall x \exists y (E(x,y) \wedge \forall z (E(y,z) \rightarrow \exists u E(z,u)))$$
   $$\psi = \exists x \forall y (E(x,y) \rightarrow \exists z (E(y,z) \wedge \forall u (E(z,u) \rightarrow \boldsymbol{F})))$$

   i. Compute $\mu_n(\varphi) + \mu_n(\psi)$. (Hint: answer is 2-3 lines max)

   ii. Compute $\lim_{n\to\infty} \mu_n(\varphi)$ for the following sentences φ:

   $$\exists x \exists y E(x,y)$$
   $$\exists x \exists y \exists z (E(x,y) \wedge E(y,z) \wedge E(z,x))$$
   $$\forall x \forall y \exists z (E(x,z) \wedge E(z,y))$$
   $$\forall x \forall y (\text{there exists a path from } x \text{ to } y)$$

   (Hint: the last sentece is not in FO, hence it is unclear a priori if $\lim \mu_n(\varphi)$ is 0 or 1, but the answer should be very simple once you solve the others.)

2 Completeness and Decidability

2. (0 points)

Consider the vocabulary consisting of the constant 0 and the function succ. Let Σ be the following set of axioms:

$\forall x \forall y(\texttt{succ}(x) = \texttt{succ}(y) \to x = y)$	injective
$\forall x(\texttt{succ}(x) \neq 0)$	
$\forall x(x \neq 0 \to \exists y(\texttt{succ}(y) = x)$	almost surjective
$\forall x(\texttt{succ}(x) \neq x)$	no cycle of length n, forall $n \geq 1$
$\forall x(\texttt{succ}(\texttt{succ}(x)) \neq x)$	
$\forall x(\texttt{succ}(\texttt{succ}(\texttt{succ}(x))) \neq x)$	
$\dots$	

(a) Prove that Σ is complete, i.e. for any sentence φ, either $\Sigma \models \varphi$ or $\Sigma \models \neg\varphi$.
(Hint: $(\mathbb{N}, 0, \texttt{succ})$ where $\texttt{succ}(x) = x + 1$ is indeed a model of Σ, but it is not the only countable model! So you need to try a bit harder.)

(b) Let $\mathbb{N}_S = (\mathbb{N}, 0, \texttt{succ})$ be the structure of *natural numbers with 0 and successor*. Prove that $\text{Th}(\mathbb{N}_S)$ is decidable. (Hint: should be very easy after you solve the previous item.)