Finite Model Theory – Homework 3

April 27, 2018

1 The Satisfiability Problem

1. (0 points)

   (a) Let $\sigma = \{U_1, \ldots, U_m\}$ be a relational vocabulary with k unary predicate symbols. Prove that sentences over this vocabulary satisfy the small model property: if φ has a model, then it has a finite model of size $\leq f(|\varphi|)$, for some computable function f. (Hint: given any infinite structure $\boldsymbol{A}$ and a number k describe a finite model $\boldsymbol{B}$ s.t. $\boldsymbol{A} \sim_k \boldsymbol{B}$.)

   (b) Prove that the satisfiability problem for a relational vocabulary consisting of only unary predicate symbols is decidable.

   (c) Let $\varphi(x)$ be formula with a free variable x, and R be a unary relational symbol. We say that φ is monotone in a relational symbol R if for any two structures $\boldsymbol{A}, \boldsymbol{B}$ with the same domain and satisfying $R^A \subseteq R^B$, and $S^A = S^B$ for every other relational symbol S, we have $\{a \in A \mid \boldsymbol{A} \models \varphi(a)\} \subseteq \{b \in B \mid \boldsymbol{B} \models \varphi(b)\}$. (Note: this is the semantic property needed for the least fixpoint, $[\mathtt{lfp}_{R,x}\varphi]$.) Prove that, if the vocabulary includes at least one binary relational symbol other than R, then the problem "given φ check if it is monotone in R over all finite structures" is undecidable.

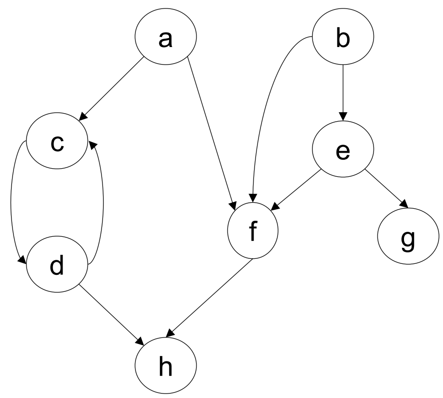
2 Descriptive Complexity

2. (0 points)

(a) Let $G = (V, E)$ be a finite graph, and consider the following query:

$$q(x) = [\mathtt{lfp}_{S,x}(\forall y(E(x, y) \rightarrow S(y)))](x)$$

i. Which nodes x does the query return on the graph below?



ii. Write an FO sentence (without fixpoints!) that is equivalent to $\forall x \neg q(x)$.

iii. Consider these complexity classes: AC^0, $PTIME$, NP, $PSPACE$. Indicate the lowest complexity class to which q belongs. You can just indicate the lowest complexity class, no need to prove that it's not lower than that (but you are welcome to do so).

(b) Consider the vocabulary $(<, P_a, P_b, P_c)$ of strings over the alphabet $\Sigma = \{a, b, c\}$.

i. Write each of the regular expressions below in FO or in MSO. Use succ, $\leq$, $\min, \max$ when needed, since these are expressible using $<$.

$$E_1 = (a|b)^*.c^* \qquad E_2 = (a.b)^* \qquad E_3 = (a.a.a)^*$$

ii. Write a regular expression describing the following language:

$$\forall S(\exists x(S(x) \wedge P_a(x))) \rightarrow (\exists y(S(y) \wedge P_b(y))$$