Algorithmic Complexity I

CSE 120 Spring 2017

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Why UPS drivers don't turn left and you probably shouldn't either

UPS solves versions of the vehicle routing problem. In these mathematical problems, you are given a set of points and the distances between them, and you have to find the best route(s) to travel through all of them.

UPS have designed their vehicle routing software to eliminate as many left-hand turns as possible. Typically, only 10% of the turns are left turns. As a result, the company claims it uses 10m gallons less fuel, emits 20,000 tonnes less carbon dioxide and delivers 350,000 more packages every year.

- https://theconversation.com/why-ups-drivers-dont-turn-left-and-you-probably-shouldnt-either-71432

Administrivia

- Assignments:
  - Binary Practice (4/21)
  - Creativity Assignment (4/24)

- Midterm in class on Wednesday, 4/26
  - 1 sheet of notes (2-sided, letter, handwritten)
  - Fill-in-the-blank(s), short answer questions, maybe simple drawing
    - Questions will cover lectures, assignments, and readings
  - Midterm Review sheet will be released tonight (4/19), will be covered in lab next week (4/25)

Outline

- **Algorithm Analysis: The Basics**
- Comparing Algorithms
- Orders of Growth

Algorithm Correctness

- An algorithm is considered correct if for every input, it reports the correct output and doesn't run forever or cause an error

- Incorrect algorithms may run forever, crash, or not return the correct answer
  - But they could still be useful!
  - *e.g.* an approximation algorithm

- Showing correctness
  - Mathematical proofs for algorithms
  - Empirical verification of implementations ← today's lecture

Algorithm Analysis

- One commonly used criterion for analyzing algorithms is computation time
  - How long does the algorithm take to run and finish its task?
  - Can be used to compare different algorithms for the same computational problem

- How to measure this time?
  - Counting in my head
  - Stopwatch
  - Within your program

Aside: Computation Time

- Computers take time to complete their tasks
  - Under the hood, it's sort of like a bunch of buckets of water filling up – you have to wait for water to reach the top of a bucket for a single computation to complete
  - Buckets take about a billionth of a second to fill (~ 1 nanosecond)
    - There are billions of them on a single chip!
- A CPU can generally only execute one instruction at a time

Timing in Processing

- The function **`millis()`** returns the number of milliseconds since starting your program (as an `int`)
  - To start timing, call and store the value in a variable
  - Call again after your computation and subtract the values

```
void draw() {
  int time = millis();
  someComputation();
  println("Took " + (millis()-time) + " milliseconds to compute.");
  noLoop();
}
```

start

end

Outline

- Algorithm Analysis: The Basics
- **Comparing Algorithms**
- Orders of Growth

Algorithm: Searching A Sorted Array

- Input: Numbers in a sorted array, desired number
- Output: If desired number is in the array (`true`/`false`)

- Algorithm 1:
  - Check each index starting from 0 for desired number
    - If equal, then report `true`
    - If not equal, then move to next index
    - If at end of array, then report `false`
  - Called Linear Search (also works for unsorted array)

```
boolean linearSearch(int num) {
  for(int i = 0; i < intArr.length; i = i + 1) {
    if(intArr[i] == num) {
      return true;
    }
  }
  return false;
}
```

Algorithm: Searching A Sorted Array

- Input: Numbers in a sorted array, desired number
- Output: If desired number is in the array (`true`/`false`)

- Algorithm 2:
  - Check "middle" index for desired number
    - If equal, then report `true`
    - If less than desired number, check *halfway* forwards next
    - If greater than desired number, check halfway backwards next
  - If no halfway point left, then report `false`
  - Called Binary Search
    - http://www.cs.armstrong.edu/liang/animation/web/BinarySearch.html

Peer Instruction Question

- On average, which algorithm would take less time to complete a search?
  - Vote at http://PollEv.com/justinh

A. Algorithm 1 (Linear Search)

B. Algorithm 2 (Binary Search) how do you "prove" this?

C. They'd take about the same amount of time

Measuring Linear Search

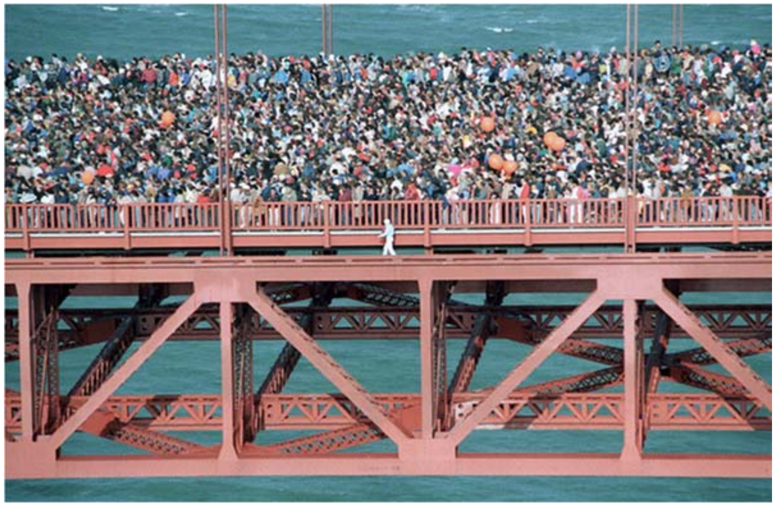
- Let's time Linear Search:

```
void draw() {
  int n = 3;
  println("Is " + n + " in intArr?");
  int time = millis();
  println(linearSearch(n));
  println("Took " + (millis()-time) + " milliseconds to compute.");
  noLoop();
}
```

- One issue: our algorithm seems to be too fast to measure! (keeps showing 0 milliseconds)
  - How can we fix this? Try longer arrays

Best Case vs. Worst Case vs. Average Case

- We were measuring close to the best case! (3 always at front of array)
  - Didn't matter how long our array was
- Could measure average case instead
  - Run many times on random numbers and average results
- Instead, we'll do worst case analysis. Why?
  - Nice to know the most time we'd ever spend
  - Worst case happens often
  - Average case is often similar to worst case



Example of Worst Case in Action

- Many web servers out there run something called "The Apache HTTP Server" (or just Apache for short)
  - When a user enters a particular URL, Apache delivers the correct files from the server to the person on the internet
  - An old version of Apache had a bug where if you entered a URL with tons of consecutive slashes, it could take *hours* to complete the request
    - *e.g.* http://someurl.com////////////////////

- Bottom line: an algorithm is often judged by its worst case behavior

What is the Worst Case?

- Discuss with your neighbor (no voting):
  - Assume `intArr.length` is `1000000` and `intArr[i] = i;`
  - What is a worst case argument for `num` for Linear Search?

    roughly the same between <not in array> and <last element of array>
  - What is a worst case argument for `num` for Binary Search?

    roughly the same between <not in array> and <ends of array>

A. 1

B. 500000

C. 1000000

D. 1000001

E. Something else

```
boolean linearSearch(int num) {
  for(int i = 0; i < intArr.length; i = i + 1) {
    if(intArr[i] == num) {
      return true;
    }
  }
  return false;
}
```

Timing Experiments

(for array length 100,000,000)
searching for 100,000,001

- Let's try running Linear Search on a worst case argument value
  - Results: 54, 31, 53, 34 ms

- Now let's run Binary Search on a worst case argument value
  - Results: 0, 0, 0, 0 ms

Runtime Intuition

- Does it seem reasonable that the runtimes were inconsistent?

- Some reasons:
  - Your computer isn't just running Processing – there's a lot of other stuff running (*e.g.* operating system, web browser)
  - The computer hardware does lots of fancy stuff to avoid slowdown due to physical limitations
    - These may not work as well each execution based on other stuff going on in your computer at the time

Empirical Analysis Conclusion

- We've shown that Binary Search is seemingly much faster than Linear Search
  - Similar to having two sprinters race each other

- Limitations:
  - Different computers may have different runtimes
  - Same computer may have different runtime on same input
  - Need to implement the algorithm in order to run it

- Goal: come up with a "universal algorithmic classifier"
  - Analogous to coming up with a metric to compare all athletes (or fighters)

Outline

- Algorithm Analysis: The Basics
- Comparing Algorithms
- **Orders of Growth**

Characterizing Algorithms

- The method computer scientists use is roughly:

1) Measure the algorithm's runtime on many different input sizes N (*e.g.* arrays of length 100, 200, 400, 800, …)
   - To avoid runtime issues, can also count number of "steps" involved

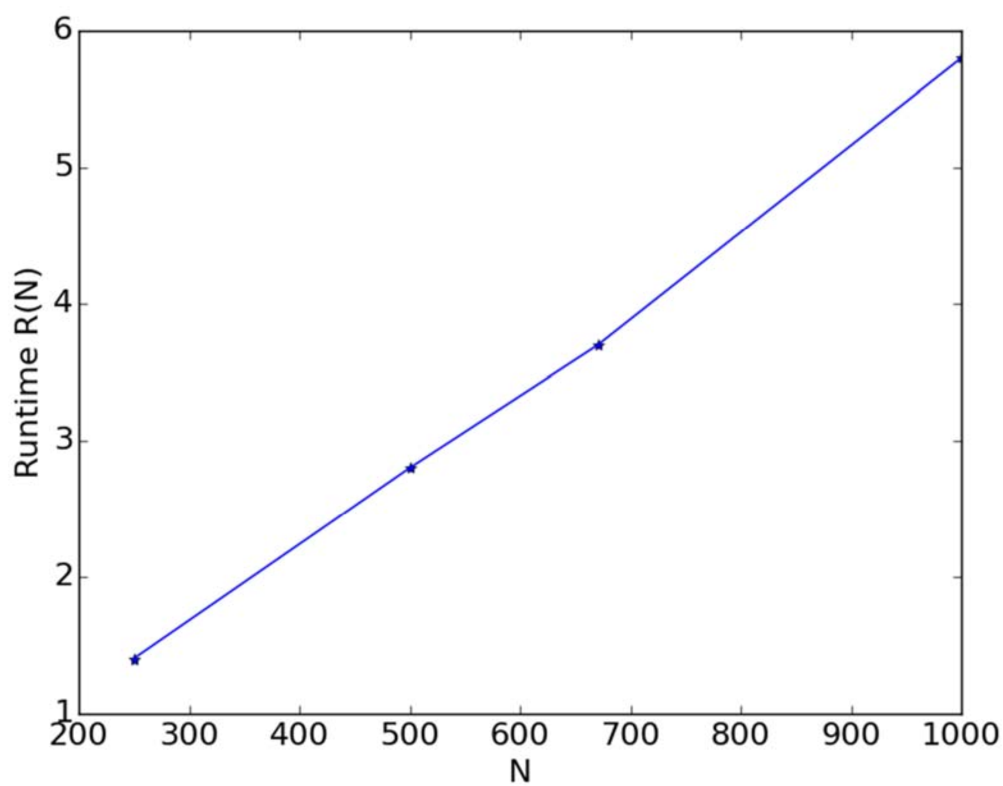
2) Make a plot of the runtime as a function of N, which we'll call $\mathrm{R(N)}$

3) Determine the general *shape* of $\mathrm{R(N)}$
   - Does $\mathrm{R(N)}$ look like N (linear), N^2 (quadratic), N^3 (cubic), $\log \mathrm{N}$ (logarithmic), etc.

Linear Search

- As the name implies, Linear Search is linear
  - If you double N, then R(N) should roughly double

	N (input size)	R(N) (time)	
N	250 items	1.4 sec	R
2N	500 items	2.8 sec	2R
	671 items	3.8 sec	
4N	1000 items	5.7 sec	≈4R

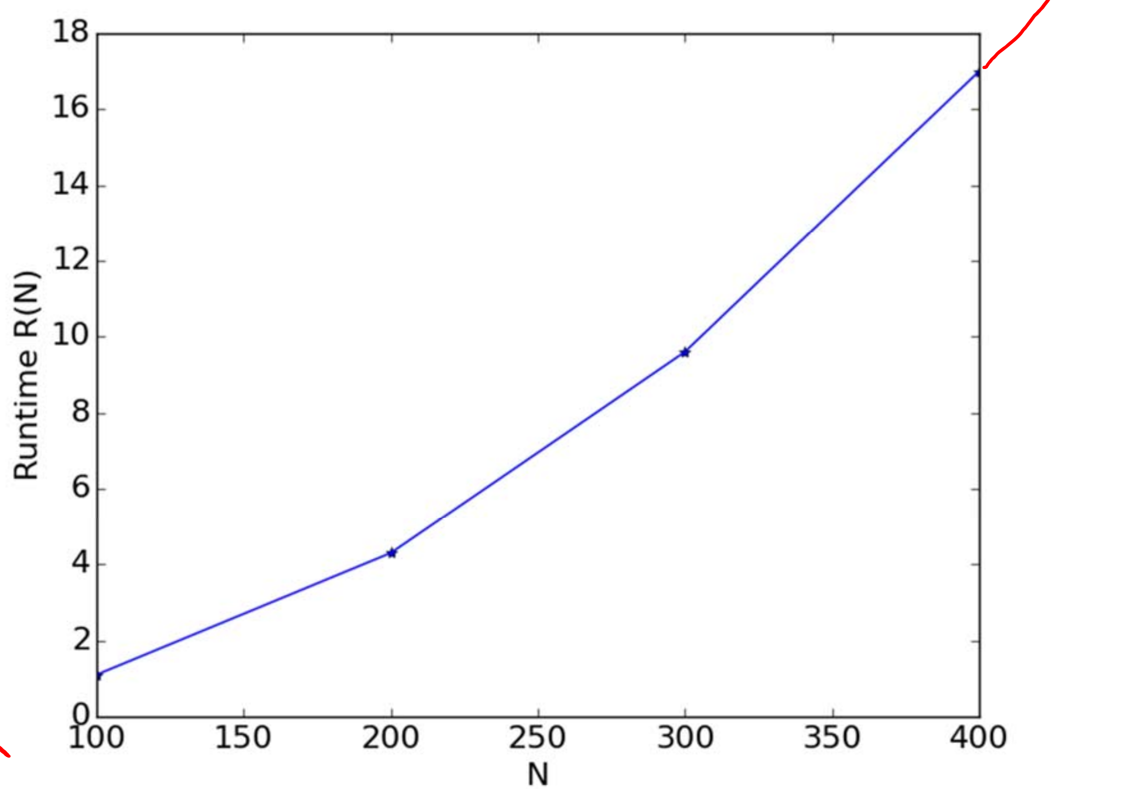


Peer Instruction Question

- Algorithm for: do any pairs in array sum to zero?
- Which function does R(N) look like?
  - Vote at http://PollEv.com/justinh

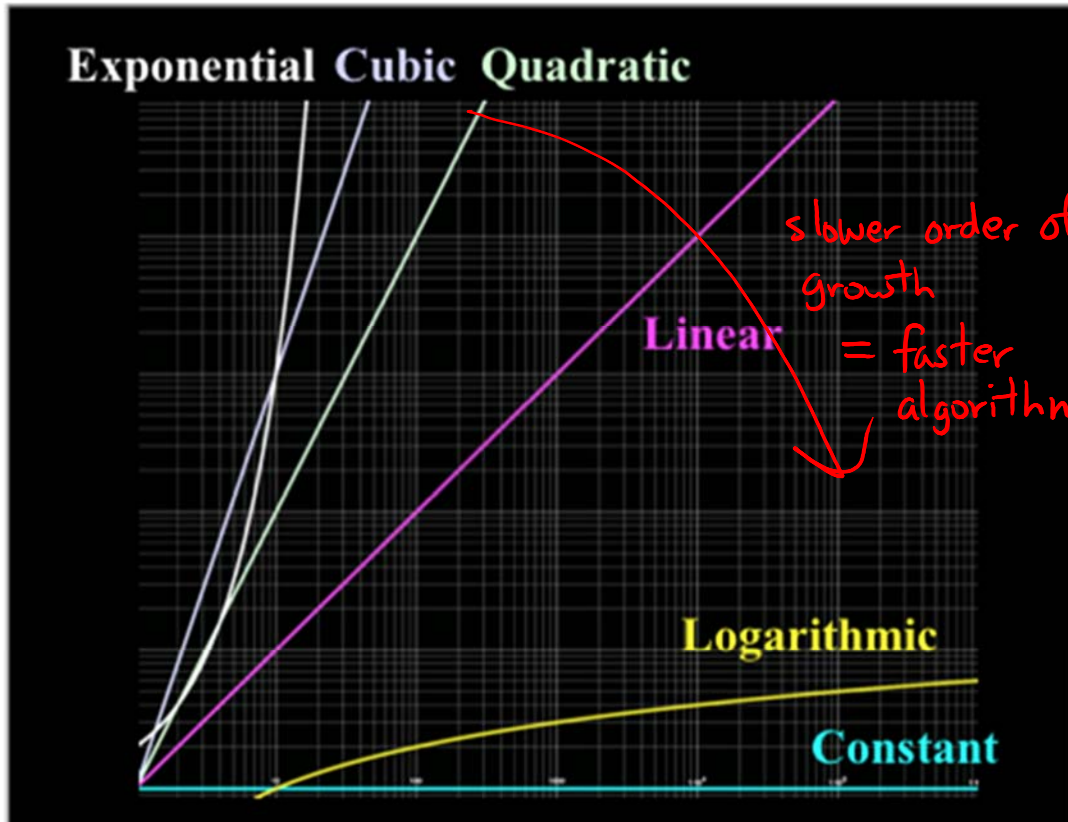
A. sqrt(N)
B. log(N)
C. N
D. $\boxed{N^2}$
E. 2^N

	N (input size)	R(N) (time)	
N	100 items	1.1 seconds	R
2N	200 items	4.3 seconds	≈4R
3N	300 items	9.6 seconds	≈9R
4N	400 items	17.0 seconds	≈16R



Orders of Growth

- The order of growth of $R(N)$ is its general shape:
  - Constant 1
  - Logarithmic $\log N$
  - Linear N
  - Quadratic N^2
  - Cubic N^3
  - Exponential 2^N
  - Factorial $N!$

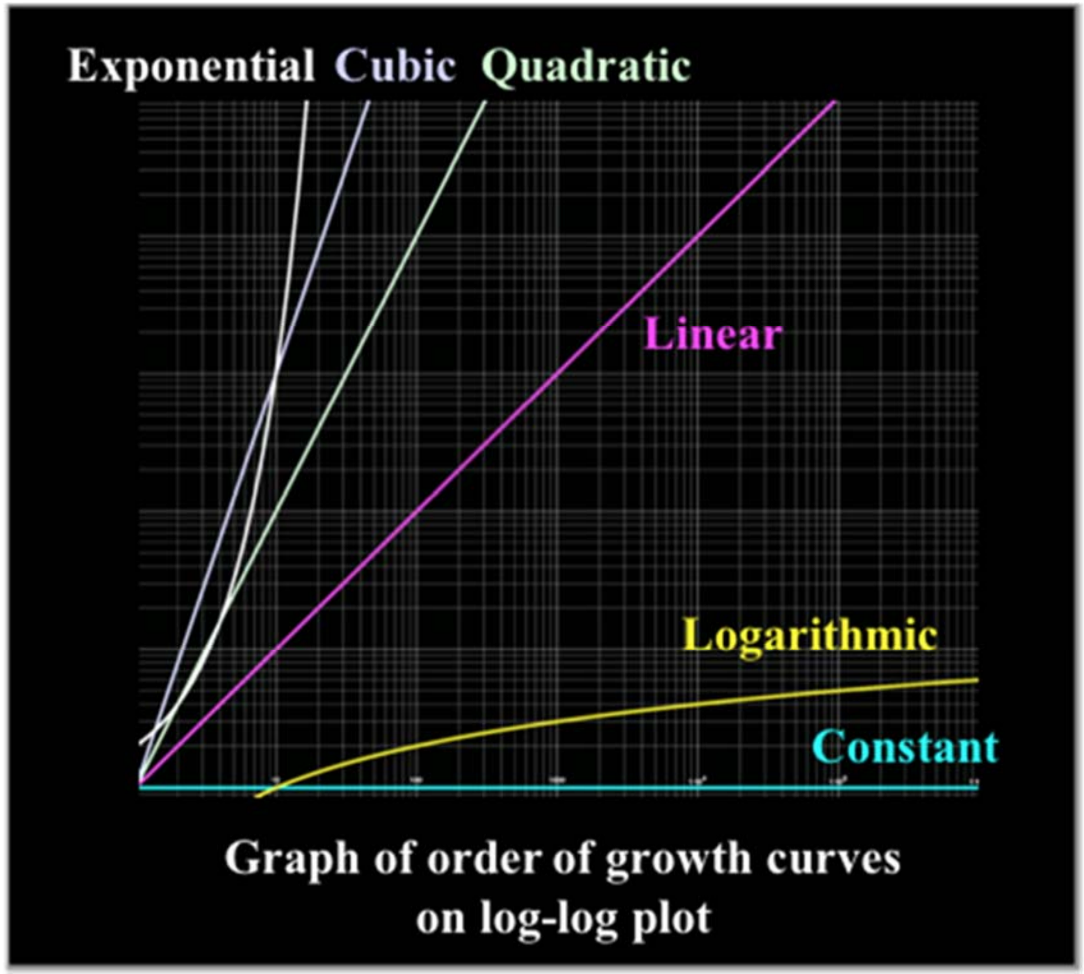


Graph of order of growth curves on log-log plot

Orders of Growth

- The order of growth of $\mathrm{R(N)}$ is its general shape:
  - Use *dominant* term
  - *e.g.* $10\mathrm{N}^2 + 4\log\mathrm{N}$ is quadratic

grows faster than 4 logN as N→∞



Graph of order of growth curves on log-log plot

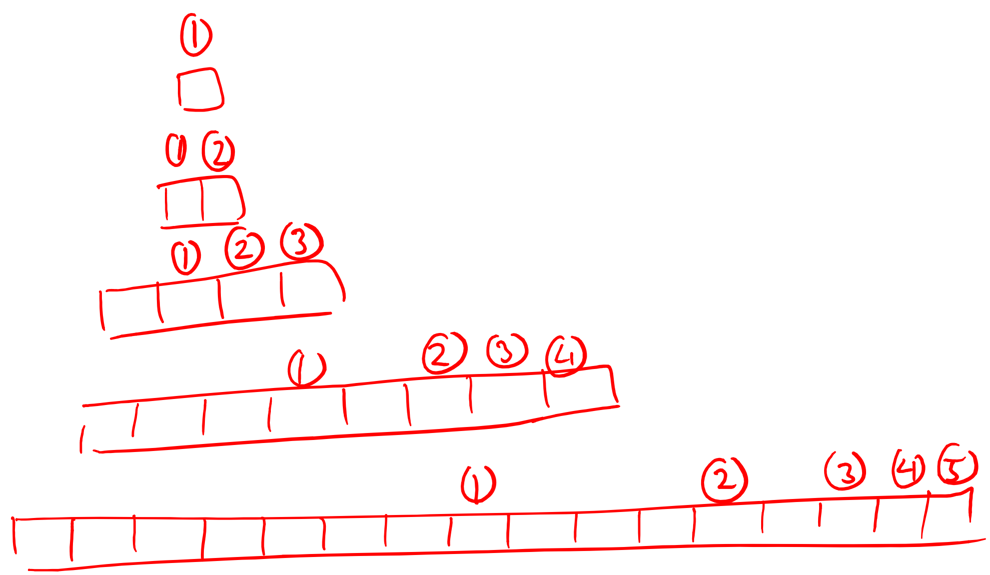
Binary Search

- What order of growth is Binary Search?
  - Analyze using number of "steps" in worst case

N (input size)	Indices to Check
1 items	1
2 items	2
4 items	3
8 items	4
16 items	5

32 items 6

64 items 7



$\boxed{R(N) = \log_2(N)}$ logarithmic!

Which is Faster?

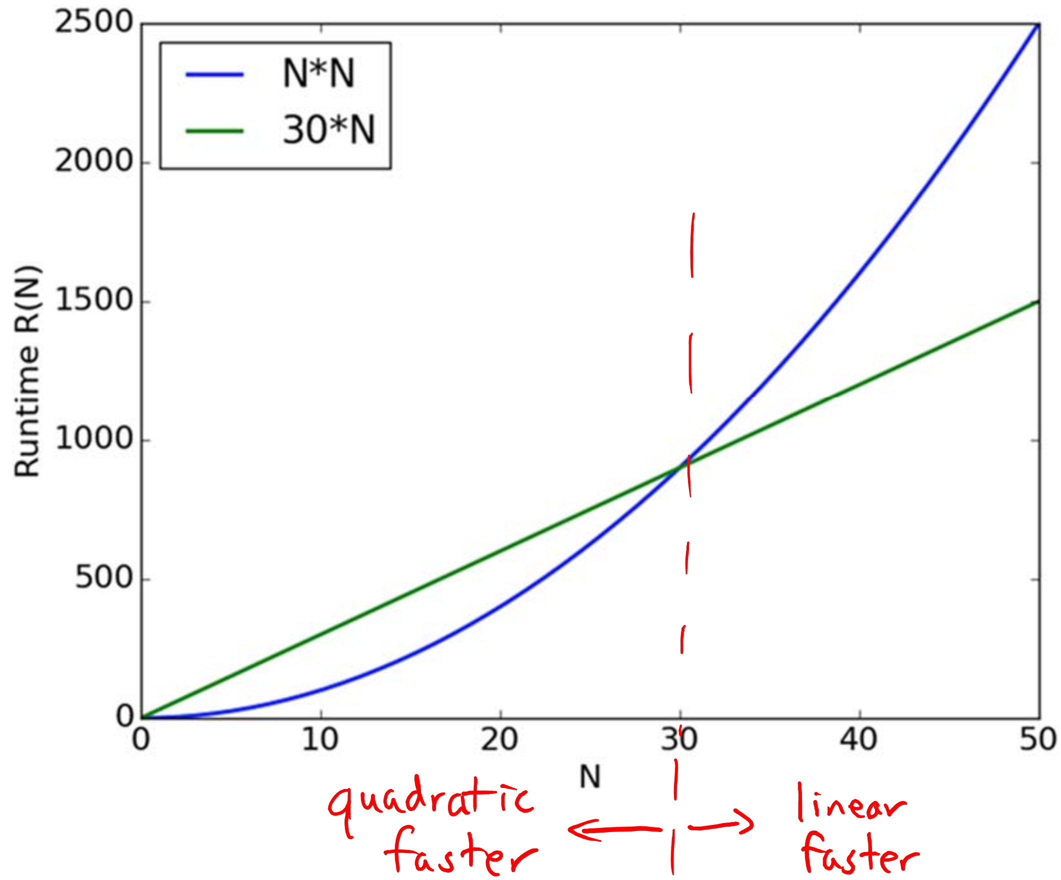
- Suppose we have two algorithms: one is linear in N and the other is quadratic in N
  - No voting

- Which is faster?

  A. Linear Algorithm

  B. Quadratic Algorithm

  C. It depends



The Reason Order of Growth Matters

- Roughly speaking, we care about really big N in real world applications
  - *e.g.* For Facebook, N (users) is ~ 1 billion
    - Want to generate list of suggested friends? Better be a fast algorithm as a function of N

- Order of growth is just a rough rule of thumb
  - There are limited cases where an algorithm with a worse order of growth can actually be faster
  - In almost all cases, order of growth works very well as a representation of an algorithm's speed

Orders of Growth Comparison

- The numbers below are rough estimates for a "typical" algorithm on a "typical" computer – provides a qualitative difference between the orders of growth

	Linear	Linearithmic	Quadratic	Cubic	Exponential	Exponential	Factorial
	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.