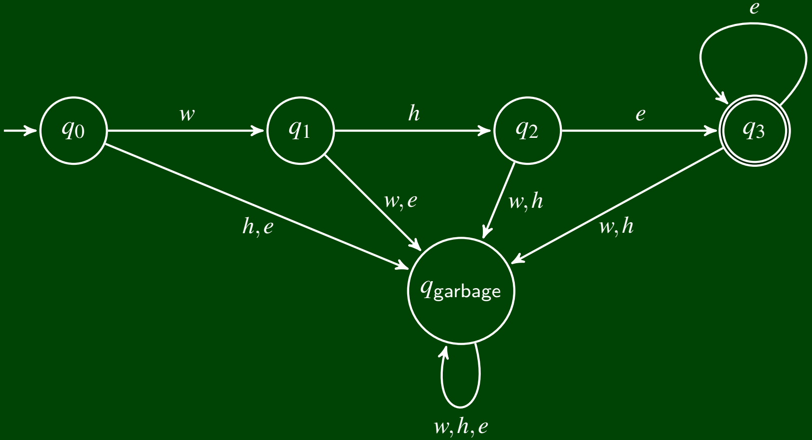
CSE 120

Computer Science Principles

Proofs & Computation



At the very "lowest" level is hardware which Justin has talked about.

At the very "highest" level is Theory which is what today is about!

In this lecture, we will explore the **abstract**! And we will apply it to **computation**!

But we start simple...

How many numbers are there?

In this lecture, we will explore the **abstract**! And we will apply it to **computation**!

But we start simple...

How many numbers are there?

...Infinity, of course!

What's the biggest number you can name?

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0, 1, 2, . . . , 4000000000000000, . . .

If you give me a number, I can get a bigger one by adding 1:

$$x \mapsto x + 1$$

If we collect all of these numbers together, we call the resulting set "the natural numbers".

Imagine an incredibly large (infinite, actually) index of numbers:

0:

1:

2:

3:

4:

5:

6:

7:

...

We say a set of numbers is **countable** (or the same size as the natural numbers) whenever we can **list them out**.

Even Numbers

"Obvious" Theorem

There are as many even numbers as odd numbers.

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There are as many even numbers as odd numbers.

Are there more even numbers than natural numbers?

0:

1:

2:

3:

4:

5:

6:

7:

Are there more integers than natural numbers?

0:

1:

2:

3:

4:

5:

6:

7:

8:

...

Are there more fractions than natural numbers?

0:

1:

2:

3:

4:

5:

6:

7:

8:

...

Are there more Strings than natural numbers?

Program

1. List out Strings of length 1:
   0. a
   1. b
   2. c
   3. ...
2. List out Strings of length 2:
   4. aa
   5. ab
   6. ac
   7. ...
3. List out Strings of length 3:
   7. aaa
   8. aab
   9. aac
   10. ...
4. ...

Are there more real numbers than natural numbers?

Incredibly, this is enough machinery to prove interesting results.

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Definition (Describable)

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Example

- "one"

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- "one"
- "two"

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- "π"

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- 1 is interesting because it's "$1 \times x = x$ for all x"
- 2 is interesting because it's "the smallest prime number"

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- What is the smallest uninteresting number?
- Is every interesting number describable?

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Questions

- What is the smallest uninteresting number?
- Is every interesting number describable?
- Is every real number describable?

Definition (Computable)

A number is **computable** when it can **unambiguously** printed out by some program.

Example

- 0 is interesting because `text("0", 0, 0)`
- 1 is interesting because `text("1", 0, 0)`
- π is interesting because...

Question

- Is every number computable?

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HALT(P) which returns true when P finishes and false if it doesn't.

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Hypothetically, consider what would happen if someone really smart has written a program:

HALT(P) which returns true when P finishes and false if it doesn't.

Then, we will find a program CONFUSE which will confuse the HALT program. . .

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Then, we will find a program CONFUSE which will confuse the HALT program. . . which means it doesn't work.

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Hypothetically, consider what would happen if someone really smart has written a program:

HALT(P) which returns true when P finishes and false if it doesn't.

Then, we will find a program CONFUSE which will confuse the HALT program...which means it doesn't work. So, it can't be written!

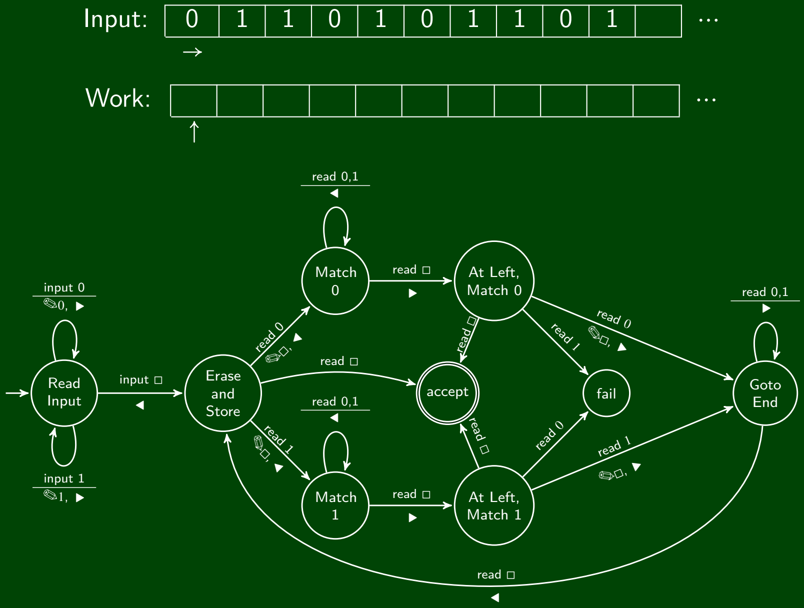
Suppose we have a program HALT such that:

HALT(P) returns true when P finishes and false if it doesn't.

Our Program

```
void CONFUSE() {
   if (HALT(SOURCE_CODE(CONFUSE))) {
      while (true) {
         text("ha ha", 0, 0);
      }
   }
   else {
      return;
   }
}
```

A Flow Chart



Input:
0 1 1 0 1 0 1 1 0 1 …
Work:
…
read 0,1
Match 0
read □
At Left, Match 0
read 0,1
input 0
✎0, ▶
Read Input
input □
Erase and Store
read 0
✎□, ▶
read □
accept
read □
read 1
read 0
✎□, ▶
fail
Goto End
input 1
✎1, ▶
read 0,1
read 1
✎□, ▶
Match 1
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Input:	0	1	1	0	1	0	1	1	0	1		⋯
	→											
Work:												⋯
	↑											

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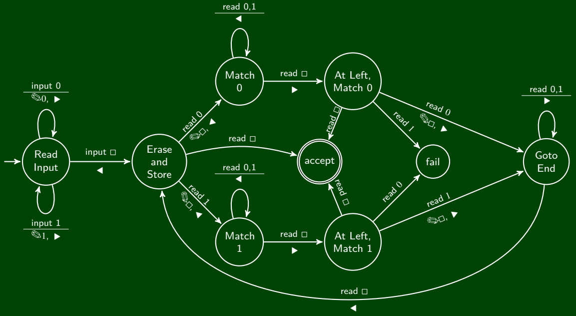
Input: | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | | …
→

Work: | | | | | | | | | | | | …
↑

A **finite-state controller**:



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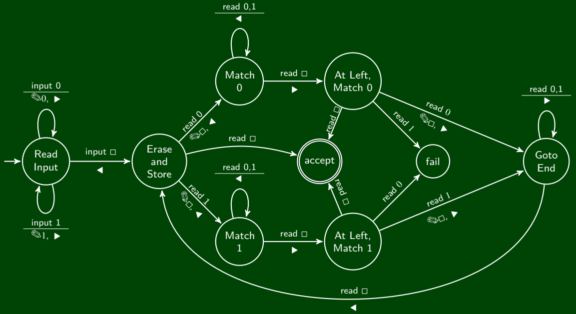

That's it. These things can decide **exactly the same languages** as register machines, and lambda calculus.