CSE 143

Program Efficiency &
Introduction to Complexity Theory

GREAT IDEAS IN COMPUTER SCIENCE

ANALYSIS OF ALGORITHMIC COMPLEXITY

Overview

- Measuring time and space used by algorithms
- Machine-independent measurements
- Costs of operations
- Asymptotic complexity – O( ) notation and complexity classes
- Comparing algorithms
- Performance tuning

Comparing Algorithms

- Example: We'll see two different list implementations
  - Dynamic expanding array
  - Linked list
- We'll see multiple ways of implementing other kinds of collections
- Which implementations are "better"?
- How do we measure?
  - Stopwatch? Why or why not?

Program Efficiency & Resources

- Goal: Find way to measure "resource" usage in a way that is independent of particular machines or implementations
- Resources
  - Execution time
  - Execution space
  - Network or disk bandwidth
  - others
- We will focus on execution time
  - Techniques/vocabulary apply to other resource measures

Example

- What is the running time of the following method?

```
// Return the sum of the elements in array.
double sum(double[ ] data) {
    double ans = 0.0;
    for (int k = 0; k < data.length; k++) {
        ans = ans + data[k];
    }
    return ans;
}
```

- How do we analyze this?
- What does the question even mean?

Analysis of Execution Time

1. First: describe the *size* of the problem in terms of one or more parameters
   - For the sum method, the size of the data array makes sense
   - Often size of data structure, but can be magnitude of some numeric parameter, etc.
2. Then, count the number of *steps* needed *as a function of the problem size*

- Need to define what a "step" is
  - First approximation: one simple statement
  - More complex statements will be multiple steps

Cost of operations: Constant Time Ops

- Constant-time operations: each take one abstract time "step"
  - Simple variable declaration/initialization (double sum = 0.0;)
  - Assignment of numeric or reference values (var = value;)
  - Arithmetic operation (+, -, *, /, %)
  - Array subscripting (a[index])
  - Simple conditional tests (x < y, p != null)
  - Operator new itself (not including constructor cost)

    Note: new takes significantly longer than simple arithmetic or assignment, but its cost is independent of the problem we're trying to analyze
- Watch out for things like method calls or constructor invocations that look simple, but can be expensive

  (because of what happens when the body of the method/constructor executes – the actual call/return operations are constant time [more below])

Cost of operations: Zero-time Ops

- Can sometimes perform operations at compile time
  - Nothing left to do at runtime
- Variable declarations without initialization

```
double[ ] overdrafts;
```

- Variable declarations with compile-time constant initializers

```
static final int maxButtons = 3;
```

- Some casts (but not those that need a runtime check)

```
int code = (int) '?';
```

- These are generally either ignored or treated as constant-time

Sequences of Statements

- Cost of

```
S1; S2; …; Sn
```

is sum of the costs of S1 + S2 + … + Sn

Conditional Statement

- We're generally trying to figure out how long it *might* take to execute a statement (*worst case*), so the cost of

```
if (condition) {
    S1;
} else {
    S2;
}
```

is usually the max cost of S1 or S2 plus cost of the condition

- Other possibilities (less common)
  - *Best case* – use the min cost of S1 or S2
  - *Expected (average) case* – probabilistic analysis needed

Analyzing Loops

- Basic analysis
  1. Calculate cost of each iteration
  2. Calculate number of iterations
  3. Total cost is the product of these

     Caution – sometimes need to add up the costs differently if the cost of each iteration is not roughly the same
- Nested loops
  - Total cost is number of iterations of the outer loop times the cost of the inner loop
  - same caution as above

Method Calls

- Cost for calling a function is cost of…

  cost of evaluating the arguments (constant or non-constant)

  \+ cost of actually calling the function (constant overhead)

  \+ cost of passing each parameter (normally constant time in Java for both numeric and reference values)

  \+ cost of executing the function body (constant or non-constant?)

  ```
  System.out.print(lineNumber);
  System.out.println("Answer is " + calculateResult(x, y*y+42.0));
  ```

- Note that "evaluating" and "passing" an argument are two different things

Exercise

- Analyze the running time of printMultTable
  - Pick the problem size
  - Count the number of steps

```
// print multiplication table with
// n rows and columns
void printMultTable(int n) {
    for (int k=1; k <= n; k++) {
        printRow(k, n);
    }
}
```

```
// print row r with length n of a
// multiplication table
void printRow(int r, int n) {
    for (int k = 1; k <= n; k++) {
        System.out.print(r*k + " ");
    }
    System.out.println( );
}
```


Analysis

Comparing Algorithms

- Suppose we analyze two algorithms and get these times (numbers of steps):
  - Algorithm 1: $37n + 2n^2 + 120$
  - Algorithm 2: $50n + 42$

  How do we compare these? What really matters?
- Answer: In the long run, the thing that is most interesting is the cost as the problem size n gets large
  - What are the costs for n=10, n=100; n=1,000; n=1,000,000?
  - Mainstream computers are so fast these days that time needed to solve small problems is rarely of interest

    Not necessarily so for slow, low-power, or embedded systems

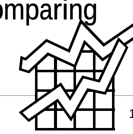
Orders of Growth

- What happens as the problem size doubles?

N	$\log_2 N$	5N	$N \log_2 N$	N^2	2^N
8	3	40	24	64	256
16	4	80	64	256	65536
32	5	160	160	1024	$\sim 10^9$
64	6	320	384	4096	$\sim 10^{19}$
128	7	640	896	16384	$\sim 10^{38}$
256	8	1280	2048	65536	$\sim 10^{76}$
10000	13	50000	10^5	10^8	$\sim 10^{3010}$

Asymptotic Complexity

- Asymptotic: Behavior of complexity function as problem size gets large
  - Only thing that really matters is higher-order term
  - Can drop low order terms and constants
- The asymptotic complexity gives us a (partial) way to answer “which algorithm is more efficient”
  - Algorithm 1: $37n + 2n^2 + 120$ is proportional to n^2
  - Algorithm 2: $50n + 42$ is proportional to n
- Graphs of functions are handy tool for comparing asymptotic behavior



Big-O Notation

- Definition: If f(n) and g(n) are two complexity functions, we say that

  $f(n) = O(g(n))$ (pronounced f(n) is O(g(n)) or is order g(n))

  if there is a constant c such that

  $f(n) \le c \cdot g(n)$

  for all sufficiently large n

Exercise 1

- Prove that 5n+3 is O(n)

Exercise 2

- Prove that $5n^2 + 42n + 17$ is $O(n^2)$

Implications

- The notation $f(n) = O(g(n))$ is *not* an equality
  (yet another abuse of the = sign; c.f., assignment operator)
- Think of it as shorthand for
  - "f(n) grows at most like g(n)" or
  - "f grows no faster than g" or
  - "f is bounded by g"
- O( ) notation is a *worst-case* analysis
  - Generally useful in practice
  - Sometimes want *average-case* or *expected-time* analysis if worst-case behavior is not typical (but often harder to analyze)

Complexity Classes

- Several common complexity classes (problem size n)
  - Constant time: $O(k)$ or $O(1)$
  - Logarithmic time: $O(\log n)$ [Base doesn't matter. Why?]
  - Linear time: $O(n)$
  - "n log n" time: $O(n \log n)$
  - Quadratic time: $O(n^2)$
  - Cubic time: $O(n^3)$

    …
  - Exponential time: $O(k^n)$
- $O(n^k)$ is often called *polynomial time*

Big-O Arithmetic

- For most common functions, comparison can be enormously simplified with a few simple rules of thumb
- Memorize complexity classes in order from smallest to largest: $O(1)$, $O(\log n)$, $O(n)$, $O(n \log n)$, $O(n^2)$, etc.
- Ignore constant factors
  $300n + 5n^4 + 6 + 2^n = O(n + n^4 + 2^n)$
- Ignore all but highest order term
  $O(n + n^4 + 2^n) = O(2^n)$

Rule of Thumb

- If the algorithm has polynomial time or better: practical
  - typical pattern: examining all data, a fixed number of times
- If the algorithm has exponential time: impractical
  - typical pattern: examine *all combinations* of data
- What to do if the algorithm is exponential?
  - Try to find a different algorithm
  - Some problems can be proved not to have a polynomial solution
  - Other problems don't have known polynomial solutions, despite years of study and effort
  - Sometimes you settle for an approximation

    The correct answer most of the time, or an almost-correct answer all of the time

Computer Science Note

- Algorithmic complexity theory is one of the key intellectual contributions of Computer Science
- Typical problems
  - What is the worst/average/best-case performance of an algorithm?
  - What is the best complexity bound for all algorithms that solve a particular problem? (i.e., how intrinsically difficult is the problem – regardless of how clever a programmer you are?)
- Interesting and (in many cases) complex, sophisticated math
  - Probabilistic and statistical as well as discrete
- Still some key open problems
  - Most notorious: P ?= NP

Analyzing List Operations (1)

- We can use O( ) notation to compare the costs of different list implementations
- Operation   Dynamic Array   Linked List
  - Construct empty list
  - Size of the list
  - isEmpty
  - clear

Analyzing List Operations (2)

- Operation   Dynamic Array   Linked List
  - Add item to end of list
  - Locate item (contains, indexOf)
  - Get an item at a given position
  - Add or remove item once it has been located

Wait! Isn't this totally bogus??

- Write better code!!
  - More clever hacking in the inner loops

    (assembly language, special-purpose hardware in extreme cases)
- Moore's law: Speeds double every 18 months
  - Wait and buy a faster computer in a year or two!

- But ...

How long is a Computer-Day?

- If a program needs f(n) microseconds to solve some problem, how big a problem can it solve in a day?
  - One day = 1,000,000*24*60*60 = $9*10^{10}$ (aprox)

f(n)	n such that f(n) = one day
n	$9 * 10^{10}$
5n	$2 * 10^{10}$
$n \log_2 n$	$3 * 10^9$
n^2	$3 * 10^5$
n^3	$4 * 10^3$
2^n	36

Speed Up The Computer by 1,000,000

- Suppose technology advances so that a future computer is 1,000,000 fast than today's

  (Or you discover a clever hack that gives a 1,000,000 speedup)

f(n)	original n	speedup on future machine
n	$9 * 10^{10}$	million times
5n	$2 * 10^{10}$	million times
$n \log_2 n$	$3 * 10^9$	60,000 times
n^2	$3 * 10^5$	1,000 times
n^3	$4 * 10^3$	100 times
2^n	36	+20

Practical Advice For Speed Lovers

- First pick the right algorithm and data structure
  - Implement it clearly and carefully, insuring correctness
- Then optimize for speed – but only where it matters
  - Constants do matter in the real world
  - Clever coding can speed things up, but the result is likely to be harder to read, modify
  - Use tools to find hotspots – concentrate on these

> **"Premature optimization is the root of all evil"**
>
> – Donald Knuth

More Advice…

"It is easier to make a correct program efficient than to make an efficient program correct"

-- Edsgar Dijkstra

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Summary

- Analyze algorithm sufficiently to determine complexity
- Compare algorithms by comparing asymptotic complexity
- For large problems, an asymptotically faster algorithm will always trump clever coding tricks
- Optimize/tune only things that actually matter, once you've picked the best algorithm

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