Name: ____

Student ID: ____

CSE 311 Winter 2011: Sample Final Exam

(closed book, closed notes except for 2-page summary)

Total: 150 points, 7 questions. Time: 1 hour and 50 minutes

Instructions:

1. Write your name and student ID on the first sheet (once you start, write your last name on all sheets). Write or mark your answers in the space provided. If you need more space or scratch paper, you can get additional sheets from the instructor or TAs. Make sure you write down the question number and your name on any additional sheets.
2. Tables for logical equivalence and set identities are included in the back.
3. Read all questions carefully before answering them. Feel free to come to the front to ask for clarifications.
4. *Hint 1*: You may answer the questions in any order, so if you find that you're having trouble with one of them, move on to another one that seems easier.
5. *Hint 2*: If you don't know the answer to a question, don't omit it - do the best you can! You may still get partial credit for whatever you wrote down. Good luck!

Do not start until you are told to do so…

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1. (25 points: 5 each) Logic, Proofs, Sets, and Functions.
Circle True (T) or False (F) below. Very briefly justify your answers (e.g., by contradiction or an example/counter-example, by citing a theorem or result we proved in class, or by *briefly* sketching a construction).

a. $p \vee (q \rightarrow r) \equiv q \rightarrow (r \vee p)$ ……………………………......................… T F
Why/Why not?

b. $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$ where the domain of each variable consists of all real numbers ………………..………………………..……….……………….. T F
Why/Why not?

c. The following argument is valid:
"If you get A's in all your CSE classes, then Google will hire you. Google hired you. Therefore, you got A's in all your CSE classes."……………………... T F
Why/Why not?

(continued on next page)

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1. (cont.)

d. For any two subsets A and B of a universal set U, $A \subseteq B \to \overline{A} \subseteq \overline{B}$ ……..….. T F
Why/Why not?

e. The function f(x,y) = x (mod y) from $Z^{+} \times Z^{+}$ to Z^{+} is a bijection…..…..… T F
Why/Why not?

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2. (25 points: 10, 5, 10 points) Number Theory

a. Suppose $n \mid m$, where m and n are integers > 1. Show that if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

b. Use the Euclidean algorithm to verify that gcd(143,311) = 1. Show the results of the successive division steps of the algorithm.

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c. Use your results in b to solve the linear congruence $143x \equiv 2 \pmod{311}$.

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3. (25 points: 10, 5, 10 points) Finite State Automata and Turing Machines

a. Let V = {0,1}. Draw the state diagram of a *deterministic* finite automaton (DFA) that recognizes the language L = {$w \in V^*$ | w begins with the substring 00 <u>or</u> ends in the substring 11}.

b. Give a regular expression for L above.

c. Consider a Turing machine M described by the five-tuples (s0,0,s0,1,R), (s0,1,s0,1,R), (s0,B,s1,B,L), (s1,1,s2,1,R) where s0 is the start state and s2 is the accept state. What does M do when given (i) 1000 as input? (ii) empty string as input?

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4. (20 points; 10 each) Induction

a. Use mathematical induction to prove that for all integers n > 1,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}<2-\frac{1}{n}$$

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b. Use strong induction to prove that you can form all dollar amounts of money $\geq \$5$ using just two-dollar bills and five-dollar bills.

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5. (15 points; 10 and 5 points) Relations

Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $a+d=b+c$.

a. Prove that R is an equivalence relation.

b. What is the equivalence class of (1,2)?

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6. (25 points: 5 each) Boolean Algebra and Circuits

Let $F(x, y, z) = xy + \overline{xz}$.

a. Give a table expressing the values of F for all possible input values.

b. Draw a 3-cube to represent F.

c. Find the sum-of-products expansion of F.

d. Express F using only the operators $\cdot$ and $\overline{}$.

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e. Draw a circuit for *F* using inverters, AND gates, and OR gates. Design your circuit from the definition of *F* and not the forms in (c) or (d).

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7. (15 points: 8, 7 points) Graphs and Trees

An intersection graph of a collection of sets is the graph that has a vertex for each set and an edge connecting the vertices representing two sets iff these sets have a nonempty intersection.

a. Draw the intersection graph for the following collection of sets:
$A_1 = \{x \mid x < 0\}$, $A_2 = \{x \mid -1 < x < 0\}$, $A_3 = \{x \mid 0 < x < 1\}$, $A_4 = \{x \mid -1 < x < 1\}$, $A_5 = \{x \mid x > -1\}$, A_6 = the set of all real numbers.

b. Write down 5 sets over the universe U = {1, 2, 3,4,5} such that their intersection graph is a tree. Draw the tree.

[End of Exam]

Have a great spring break!

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TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

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TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditionals.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	***Name***
$\dfrac{\forall x\, P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x\, P(x)}$	Existential generalization

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TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

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TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws