Name: ______________________________________
Student ID: ______________________________________

CSE 311 Winter 2011: Sample Final Exam SOLUTIONS

(closed book, closed notes except for 2-page summary)

Total: 150 points, 7 questions. Time: 1 hour and 50 minutes

Instructions:

1. Write your name and student ID on the first sheet (once you start, write your last name on all sheets). Write or mark your answers in the space provided. If you need more space or scratch paper, you can get additional sheets from the instructor or TAs. Make sure you write down the question number and your name on any additional sheets.
2. Tables for logical equivalence and set identities are included in the back.
3. Read all questions carefully before answering them. Feel free to come to the front to ask for clarifications.
4. *Hint 1*: You may answer the questions in any order, so if you find that you're having trouble with one of them, move on to another one that seems easier.
5. *Hint 2*: If you don't know the answer to a question, don't omit it - do the best you can! You may still get partial credit for whatever you wrote down. Good luck!

Do not start until you are told to do so…

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1. (25 points: 5 each) Logic, Proofs, Sets, and Functions.
Circle True (T) or False (F) below. Very briefly justify your answers (e.g., by contradiction or an example/counter-example, by citing a theorem or result we proved in class, or by *briefly* sketching a construction).

a. $p \vee (q \rightarrow r) \equiv q \rightarrow (r \vee p)$ …………………………………………… T F
Why/Why not?

TRUE. Proof:

$\mathbf{p \vee (q \rightarrow r)}$	$\mathbf{\equiv p \vee (\neg q \vee r)}$	**Table 7 equivalence**
	$\mathbf{\equiv (\neg q \vee r) \vee p}$	**Commutative law**
	$\mathbf{\equiv \neg q \vee (r \vee p)}$	**Associative law**
	$\mathbf{\equiv q \rightarrow (r \vee p)}$	**Table 7 equivalence**

b. $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$ where the domain of each variable consists of all real numbers ………………………………………………………………………….. T F
Why/Why not?

FALSE. Proof by contradiction.
Suppose true. Pick two nonzero real numbers y_1 and y_2, $y_1 \neq y_2$. Then we have $x = 1/y_1 \neq 1/y_2 = x$. Contradiction.

c. The following argument is valid:
"If you get A's in all your CSE classes, then Google will hire you. Google hired you. Therefore, you got A's in all your CSE classes."……………………... T F
Why/Why not?

FALSE. This is based on $[(p \rightarrow q)$ and $q] \rightarrow p$ which is not a tautology (see p. 69 in the text under Fallacies).

(continued on next page)

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1. (cont.)

d. For any two subsets A and B of a universal set U, $A \subseteq B \rightarrow \overline{A} \subseteq \overline{B}$ ……..….. T F
Why/Why not?

FALSE. Counterexample:
Let $U = \{1,2,3\}$, $A = \{1\}$, $B = \{1,2\}$
Then, $\overline{A} = \{2,3\}$, $\overline{B} = \{3\}$
$\overline{A}$ is not a subset of $\overline{B}$

e. The function f(x,y) = x (mod y) from $Z^{+} \times Z^{+}$ to Z^{+} is a bijection…..……. T F
Why/Why not?

FALSE. f is not 1-1, e.g., f(1,311) = 1 = f(1,312). Therefore not a bijection (note that f is onto).

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2. (25 points: 10, 5, 10 points) Number Theory

a. Suppose n | m, where m and n are integers > 1. Show that if a ≡ b (mod m), where a and b are integers, then a ≡ b (mod n).

See solution at the back of the text for Exercise 21, Section 3.4.

b. Use the Euclidean algorithm to verify that gcd(143,311) = 1. Show the results of the successive division steps of the algorithm.

311 = 143*2 + 25
143 = 25*5 + 18
25 = 18*1 + 7
18 = 7*2 + 4
7 = 4*1 + 3
4 = 3*1 + 1
3 = 1*3
The last nonzero remainder is 1. Hence, gcd(143,311) = 1.

c. Use your results in b to solve the linear congruence 143x ≡ 2 (mod 311).

First find the inverse of 143 mod 311 by reversing the steps in b:
1 = 4 – 3*1 = 4 – (7 – 4*1) = 4*2 – 7 = (18-7*2)*2 - 7
= 18*2 – 7*5 = 18*2-(25-18*1)*5 = 18*7-25*5
= (143-25*5)*7-25*5 = 143*7-25*40 = 143*7-(311-143*2)*40
= 143*87-311*40
Therefore, inverse of 143 mod 311 is 87.
Multiplying both sides of the congruence by 87, we get:
143*87x ≡ 2*87 (mod 311)
x ≡ 174 (mod 311)

Check this solution by substituting for x in the congruence:

(143*174) mod 311 = 2

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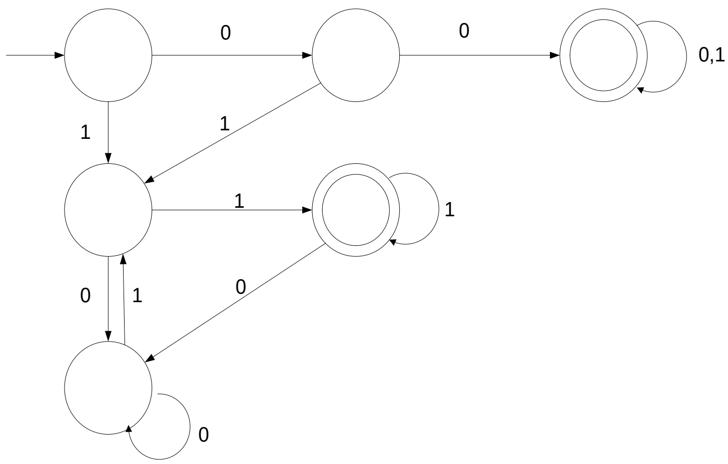
3. (25 points: 10, 5, 10 points) Finite State Automata and Turing Machines

a. Let V = {0,1}. Draw the state diagram of a *deterministic* finite automaton (DFA) that recognizes the language L = {$w \in V^*$ | w begins with the substring 00 <u>or</u> ends in the substring 11}.



b. Give a regular expression for L above.

$00(0 \cup 1)^* \cup (0 \cup 1)^*11$

c. Consider a Turing machine M described by the five-tuples (s0,0,s0,1,R), (s0,1,s0,1,R), (s0,B,s1,B,L), (s1,1,s2,1,R) where s0 is the start state and s2 is the accept state. What does M do when given (i) 1000 as input? (ii) empty string as input?

(i) M changes each 0 in the input to a 1 and leaves 1's unchanged. So for 1000, M changes the tape to 1111, enters state s2, and therefore accepts the input.

(ii) For the empty string input, the tape contains all blanks, so M goes from s0 to s1. Since there is no five-tuple for s1 reading a blank, the machine halts in the state s1 and therefore rejects the input.

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4. (20 points; 10 each) Induction

a. Use mathematical induction to prove that for all integers n > 1,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<2-\frac{1}{n}$$

See solution at the back of the text for Exercise 19, Section 4.1.

b. Use strong induction to prove that you can form all dollar amounts of money ≥ $5 using just two-dollar bills and five-dollar bills.

See solution at the back of the text for Exercise 7, Section 4.2.

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5. (15 points; 10 and 5 points) Relations

Let R be the relation on the set of ordered pairs of positive integers such that ((a,b),(c,d)) $\in$R if and only if a+d=b+c.

a. Prove that R is an equivalence relation.

See solution at the back of the text for Exercise 15, Section 8.5. Use the definition of equivalence relation and not a result such as Exercise 9 in Sec. 8.5.

b. What is the equivalence class of (1,2)?

See solution at the back of the text for Exercise 39, Section 8.5.

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6. (25 points: 5 each) Boolean Algebra and Circuits

Let $F(x, y, z) = xy + \overline{xz}$.

a. Give a table expressing the values of *F* for all possible input values.

x	y	z	xy	xz	$\overline{xz}$	$xy + \overline{xz}$
1	**1**	**1**	**1**	**1**	**0**	**1**
1	**1**	**0**	**1**	**0**	**1**	**1**
1	**0**	**1**	**0**	**1**	**0**	**0**
1	**0**	**0**	**0**	**0**	**1**	**1**
0	**1**	**1**	**0**	**0**	**1**	**1**
0	**1**	**0**	**0**	**0**	**1**	**1**
0	**0**	**1**	**0**	**0**	**1**	**1**
0	**0**	**0**	**0**	**0**	**1**	**1**

b. Draw a 3-cube to represent *F*.

From table above, construct cube as in Figure 1 in Section 11.1.

c. Find the sum-of-products expansion of *F*.

Construct from table:

$F(x, y, z) = xyz + xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

d. Express *F* using only the operators $\cdot$ and $\bar{\ }$.

Using DeMorgan's law and negating both sides, we get $a + b = \overline{\bar{a}\bar{b}}$ **. We can apply this to the definition of** ***F*** **to get:**

$F(x, y, z) = xy + \overline{xz} = \overline{(xy) \cdot (\overline{xz})}$

e. Draw a circuit for *F* using inverters, AND gates, and OR gates. Design your circuit from the definition of *F* and not the forms in (c) or (d).

Follow the examples in the lecture slides or Section 11.3.

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7. (15 points: 8, 7 points) Graphs and Trees

An intersection graph of a collection of sets is the graph that has a vertex for each set and an edge connecting the vertices representing two sets iff these sets have a nonempty intersection.

a. Draw the intersection graph for the following collection of sets:
$A_1 = \{x \mid x < 0\}$, $A_2 = \{x \mid -1 < x < 0\}$, $A_3 = \{x \mid 0 < x < 1\}$, $A_4 = \{x \mid -1 < x < 1\}$, $A_5 = \{x \mid x > -1\}$, A_6 = the set of all real numbers.

See solution at the back of the text for Exercise 13c, Section 9.1.

b. Write down 5 sets over the universe U = {1, 2, 3,4,5} such that their intersection graph is a tree. Draw the tree.

Several solutions are possible. One solution:
$A_1 = \{1, 3\}$, $A_2 = \{1, 2\}$, $A_3 = \{3,4,5\}$, $A_4 = \{4\}$, $A_5 = \{5\}$

```
      A1
     /  \
   A2    A3
        /  \
      A4    A5
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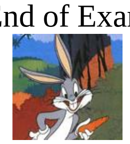
[End of Exam]



Have a great spring break!