Relations

Sections 8.1 & 8.5

Relations and Their Properties

A binary relation R from set A to set B is a subset of Cartesian product $A \times B$

Example: $A = \text{UW students}$ $B = \text{UW courses}$

$$R = \{(a,b) \mid a \text{ is enrolled in } b\}$$

Example: $A = \{0,1,2\}$ $B = \{a,b\}$

$$R = \{(0,a),(0,b),(1,a),(2,b)\}$$

A relation on set A is a subset of $A \times A$

Example:

A relation on set $A = \{1,2,3,4\}$:

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$

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More Examples

Relations over integers:

$$R = \{(a,b) \mid a > b\}$$

$$R = \{(a,b) \mid a = b \text{ or } a = -b\}$$

$$R = \{(a,b) \mid a \equiv b (\bmod m)\} \text{ for positive integer } m > 1$$

$$R = \{(a,b) \mid b = a + 1\}$$

(Actually a function)

Functions as Relations

$R = \{(a,b) \mid b = a+1\}$ Relation over integers Z

$f(a) = b = a+1$ Function from Z to Z

$f : Z \to Z$

Function from A to B assigns exactly one element from B to each input from A

i.e., a functions is a restricted type of relation where every a in A is in exactly one ordered pair (a,b).

Reflexive relation R on set A :

$$\forall a \in A, \quad (a,a) \in R$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(3,3),(4,3),(4,4)\}$$

Symmetric relation R :

$$(a,b)\in R \rightarrow (b,a)\in R$$

Example: $A=\{1,2,3,4\}$

$$R=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(4,4)\}$$

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Antisymmetric relation R :

$$(a,b)\in R \wedge (b,a)\in R \rightarrow a=b$$

Example: $A=\{1,2,3,4\}$

$$R=\{(1,1),(1,2),(2,2),(3,4),(4,4)\}$$

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Transitive relation R :

$$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$$

Example: $A = \{1,2,3,4\}$

$$R = \{(1,1),(1,2),(2,3),(3,4)(1,3),(1,4),(2,4)\}$$

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Combining Relations

$$R_1 = \{(1,1),(2,2),(3,3)\}$$
$$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$$

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$
$$R_1 \cap R_2 = \{(1,1)\}$$
$$R_1 - R_2 = \{(2,2),(3,3))$$

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Composite relation: $S \circ R$

$$(a,b) \in S \circ R \leftrightarrow \exists x : (a,x) \in R \wedge (x,b) \in S$$

Note: $(a,b) \in R \wedge (b,c) \in S \rightarrow (a,c) \in S \circ R$

Example:

$$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$$

$$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$$

$$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

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Power of relation: R^n

$$R^1 = R \qquad R^{n+1} = R^n \circ R$$

Example: $R = \{(1,1),(2,1),(3,2),(4,3)\}$

$$R^2 = R \circ R = \{(1,1),(2,1),(3,1)(4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1)(4,1)\}$$

$$R^4 = R^3 \circ R = R^3$$

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Theorem: A relation R is transitive if and only if $R^n \subseteq R$ for all $n = 1,2,3,\ldots$

Proof: 1. If part: $R^2 \subseteq R$

2. Only if part: use induction

1. If part: We will show that if $R^2 \subseteq R$ then R is transitive

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Assumption: $R^2 \subseteq R$

Definition of power: $R^2 = R \circ R$

Definition of composition:

$$(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R \circ R$$

$\Rightarrow (a,c) \in R$

Therefore, R is transitive

2. Only if part:

We will show that if R is transitive then $R^n \subseteq R$ for all $n \geq 1$

Proof by induction on n

Inductive basis: $n = 1$

It trivially holds $R^1 = R \subseteq R$

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Inductive hypothesis:

Assume that $R^k \subseteq R$

for all $1 \leq k \leq n$

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Inductive step: We will prove $R^{n+1} \subseteq R$

Take arbitrary $(a,b) \in R^{n+1}$

We will show $(a,b) \in R$

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$$(a,b) \in R^{n+1}$$

⇓ definition of power

$$(a,b) \in R^n \circ R$$

⇓ definition of composition

$$\exists x : (a,x) \in R \wedge (x,b) \in R^n$$

⇓ inductive hypothesis $R^n \subseteq R$

$$\exists x : (a,x) \in R \wedge (x,b) \in R$$

⇓ R is transitive

$$(a,b) \in R$$

End of Proof

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