Section 3 Worksheet Solutions

April 18, 2013

Proof with Even and Odd

Premises: $\forall x \neg(Even(x) \wedge Odd(x))$, $\forall x(Even(x) \rightarrow Even(x^2))$, $\forall x(Even(x) \vee Odd(x))$

Conclusion: $\forall x(Odd(x^2) \rightarrow Odd(x))$

Proof with Even and Odd

(1)	$\forall x(\text{Even}(x) \rightarrow \text{Even}(x^2))$	Premise
(2)	$\text{Even}(a) \rightarrow \text{Even}(a^2)$	$\forall$ elim., a arbitrary
(3)	$\neg\text{Even}(a^2) \rightarrow \neg\text{Even}(a)$	Contrapositive of (2)
(4)	$\forall x(\text{Odd}(x) \vee \text{Even}(x))$	Premise
(5)	$\text{Odd}(a) \vee \text{Even}(a)$	$\forall$ elim.
(6)	$\neg\text{Even}(a)$, $\therefore$ $\text{Odd}(a)$	Disjunctive Syllogism
(7)	$\text{Odd}(a^2) \rightarrow \text{Odd}(a)$	Applying (6) to (3) (similar argument for a^2)
(8)	$\forall x(\text{Odd}(x^2) \rightarrow \text{Odd}(x))$	$\forall$ intro.

Proof Practice with Predicate Logic

1. Premises: $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$, $\forall x(P(x) \wedge R(x))$

Conclusion: $\forall x(R(x) \wedge S(x))$

(1) $\forall x(P(x) \wedge R(x))$	Premise
(2) $P(a) \wedge R(a)$	$\forall$ elimination, a arb.
(3) $\therefore P(a)$, $R(a)$	Simplification
(4) $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$	Premise
(5) $P(a) \rightarrow (Q(a) \wedge S(a))$	$\forall$ elimination
(6) $P(a)$, $\therefore (Q(a) \wedge S(a))$	Modus ponens
(7) $\therefore Q(a)$, $S(a)$	Simplification

Proof Practice with Predicate Logic

(1) $\forall x(P(x) \wedge R(x))$	Premise
(2) $P(a) \wedge R(a)$	$\forall$ elimination, a arb.
(3) $\therefore P(a)$, $R(a)$	Simplification
(4) $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$	Premise
(5) $P(a) \rightarrow (Q(a) \wedge S(a))$	$\forall$ elimination
(6) $P(a)$, $\therefore (Q(a) \wedge S(a))$	Modus ponens
(7) $\therefore Q(a)$, $S(a)$	Simplification
(8) $R(a)$, $S(a)$ $\therefore R(a) \wedge S(a)$	Conjunction on (3) and (7)
(9) $\forall x(R(x) \wedge S(x))$	$\forall$ introduction

Proof Practice with Predicate Logic

2. Premises: $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, $\exists x \neg P(x)$

Conclusion: $\exists x \neg R(x)$

(1) $\exists x \neg P(x)$	Premise
(2) $\neg P(c)$	$\exists$ elimination, specific c
(3) $\forall x(P(x) \vee Q(x))$	Premise
(4) $P(c) \vee Q(c)$	$\forall$ elimination
(5) $\neg P(c) \wedge (P(c) \vee Q(c))$	Conjunction of (2), (4)
(6) $\therefore Q(c)$	Disjunctive Syllogism
(7) $\forall x(\neg Q(x) \vee S(x))$	Premise

Proof Practice with Predicate Logic

(5) $\neg P(c) \land (P(c) \lor Q(c))$	Conjunction of (2), (4)
(6) $\therefore Q(c)$	Disjunctive Syllogism
(7) $\forall x(\neg Q(x) \lor S(x))$	Premise
(8) $\neg Q(c) \lor S(c)$	$\forall$ elim.
(9) $Q(c) \land (\neg Q(c) \lor S(c))$	Conjunction on (6), (8)
(10) $\therefore S(c)$	Disjunctive syllogism
(11) $\forall x(R(x) \rightarrow \neg S(x))$	Premise
(12) $R(c) \rightarrow \neg S(c)$	$\forall$ elim.
(13) $S(c) \rightarrow \neg R(c)$	Contrapositive
(14) $S(c), \therefore \neg R(c)$	Modus Ponens (S(c) true from (10))

Proof Practice with Predicate Logic

(11) $\forall x(R(x) \rightarrow \neg S(x))$	Premise
(12) $R(c) \rightarrow \neg S(c)$	$\forall$ elim.
(13) $S(c) \rightarrow \neg R(c)$	Contrapositive
(14) $S(c)$, $\therefore \neg R(c)$	Modus Ponens (S(c) true from (10))
(15) $\exists x \neg R(x)$	$\exists$ introduction

Notes on Previous Slides

- Note on ∀ elimination: When eliminating ∀, you use an arbitrary value in the domain (what we call *a* in the previous proofs). Later on, when I eliminate ∀ from other premises, I pick that *same* arbitrary *a* so that I can combine the statements I've derived. You can introduce ∀ again as long as *a* is still arbitrary (that is, you haven't combined it with an expression that involved an ∃ – see next slide) .

Notes on Previous Slides

- Note on $\exists$ elimination: When eliminating $\exists$, you need to remember that you are picking a *specific* value in the domain (this is *c* in our previous proofs). Also, when I eliminate $\forall$ in part 2 of the second problem, note that I chose *c* again so that I can combine the two expressions I've derived. This is allowed since $\forall$ implies the expression is true for *every* value in the domain, including *c*. However, note that you can't introduce $\forall$ from this, since *c* is still the specific value that we chose when eliminating $\exists$.

English Proof Practice

Prove: If $n=ab$ for positive a,b, then $a\leq\sqrt{n}$ or $b\leq\sqrt{n}$.

Note: Not easy to see how to directly derive $a\leq\sqrt{n}$ or $b\leq\sqrt{n}$ from $n=ab$. Therefore we will use a proof by contraposition: $\neg Q(x) \rightarrow \neg P(x)$ instead of $P(x) \rightarrow Q(x)$.

$P(x) \rightarrow Q(x)$: ($n=ab$ for positive a,b)$\rightarrow$($a\leq\sqrt{n}$ or $b\leq\sqrt{n}$)

English Proof Practice

Proof: Assume $\neg(a\leq\sqrt{n} \vee b\leq\sqrt{n})$.

Then $a>\sqrt{n}$ and $b>\sqrt{n}$ by DeMorgan's Law.

Next, multiplying and noting that a and b are both positive (and thus our greater than sign does not get flipped), we get $ab > \sqrt{n}\sqrt{n} = n$.

So $ab>n$ and thus $ab\neq n$, which is $\neg(ab=n)$.

We have shown $\neg(a\leq\sqrt{n} \vee b\leq\sqrt{n}) \rightarrow \neg(ab=n)$, which is the contrapositive of $(ab=n) \rightarrow (a\leq\sqrt{n} \vee b\leq\sqrt{n})$, which is what we were trying to show. ■