## CSE 311 Quiz Section: June 6, 2013 (Solutions)

## 1 Countability

a) An arbitrary rational number can be expressed as $\frac{p}{q}$, where $p$ and $q$ are both integers and $q \neq 0$. As we showed in lecture, integers are countable. Therefore, we can get an enumeration covering all rational numbers by dovetailing on integers and integers:

|  | 1 | -1 | 2 | -2 | $\ldots$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $0 / 1$ | $0 /-1$ | $0 / 2$ | $0 /-2$ | $\ldots$ |
| 1 | $1 / 1$ | $1 /-1$ | $1 / 2$ | $1 /-2$ | $\ldots$ |
| -1 | $-1 / 1$ | $-1 /-1$ | $-1 / 2$ | $-1 /-2$ | $\ldots$ |
| 2 | $2 / 1$ | $2 /-1$ | $2 / 2$ | $2 /-2$ | $\ldots$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

Notice that there are multiple representations of each rational number in this table, e.g. $1 / 1=-1 /-1=2 / 2 \ldots$ This is fine, the important part is that we will include all possible rational numbers in our dovetailing.
b) First, we want to show that binary strings are countable. Order binary strings by length, then sort them lexographically. Our list will be $\lambda, 0,1,00,01,10,11,000,001,010, \ldots$ This will cover all possible binary strings, so binary strings are countable. Therefore, we can get an enumeration covering all pairs of binary strings by dovetailing on binary strings and binary strings:

|  | $\lambda$ | 0 | 1 | 00 | $\ldots$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda$ | $(\lambda, \lambda)$ | $(\lambda, 0)$ | $(\lambda, 1)$ | $(\lambda, 00)$ | $\ldots$ |
| 0 | $(0, \lambda)$ | $(0,0)$ | $(0,1)$ | $(0,00)$ | $\ldots$ |
| 1 | $(1, \lambda)$ | $(1,0)$ | $(1,1)$ | $(1,00)$ | $\ldots$ |
| 00 | $(00, \lambda)$ | $(00,0)$ | $(00,1)$ | $(00,00)$ | $\ldots$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

## 2 Computability

To show that a problem is undecidable, we can show that if we had a solution to it, we could solve the halting problem.
Assume for contradiction that we can solve INFINITE. To solve the halting problem, we need to be able to solve an arbitrary instance of it. Let $\langle P\rangle, x$ be arbitrary input to the halting problem, where $<P>$ is the code of a program $P$ and $x$ is the input to that program. For given $P$ and $x$, We will define a new program $Q_{P, x}$ taking input $y$ as follows:
run P with input x

Notice that $Q_{P, x}$ halts if and only if $P$ halts with input $x$. Furthermore, notice that $Q_{P, x}$ completely ignores its own input $y$. If $P$ halts on $x$, then $Q_{P, x}$ always halts on all input $y$, and if $P$ doens't halt on $x$, then $Q_{P, x}$ never halts on any input $y$. Since there are an infinite number of possible inputs $y$ for $Q_{P, x}$, this is equivalent to saying that $P$ halts on $x$ if and only if $Q_{P, x}$ halts on an infinite number of inputs.
By our assumption, we can solve INFINITE. Solve the instance of INFINITE with $Q_{P, x}$ as input. By the definition of INFINITE, this outputs 1 if $Q$ halts on an infinite number of inputs and 0 otherwise (i.e., Q halts for 0 input, which is a finite number). Since $Q_{P, x}$ halts on an infinite number of inputs if and only if $P$ halts on $x$, our solution to INFINITE outputs 1 if $P$ halts on $x$,
and 0 otherwise. This is a solution to the halting problem, and since we know that the halting problem is undecidable, we've reached a contradiction. Therefore, our assumption is false, and INFINITE is undecidable.

## 3 Another Computability Problem

Assume for contradiction that we can solve ONE. Let $\langle P\rangle, x$ be arbitrary input to the halting problem, where $\langle P\rangle$ is the code of a program $P$ and $x$ is the input to that program. For given $P$ and $x$, We will define a new program $Q_{P, x}$ taking input! as follows:

```
delete every print statement in <P> to create a new program P'
run P' on x
print 1
```

The runtime behavior of $P^{\prime}$ will be exactly identical to that of $P$, except that $P^{\prime}$ will not print any output. Therefore, $P^{\prime}$ halts if and only if $P$ halts. Notice that since we defined $P^{\prime}$ to never print anything, $Q_{P, x}$ prints 1 if and only if $P^{\prime}$ terminates when run on x (if $P^{\prime}$ never terminated, we would never reach the "print $1 "$ statement). Equivalently, $Q_{P, x}$ prints 1 if and only if $P$ terminates when run on $x$.
By our assumption, we can solve ONE. Solve the instance of ONE with $Q_{P, x}$ as input. This outputs a 1 if and only if $P$ halts when run on $x$. This is a solution to the halting problem, and since we know that the halting problem is undecidable, we've reached a contradiction. Therefore, our assumption is false, and ONE is undecidable.

