CSE 311 Quiz Section: May 2, 2013 (Solutions)

Proofs by Induction

Say we want to prove that some proposition $P(n)$ is true for each integer $n \geq 1$. A proof by induction consists of five steps:

(1) State what we want to prove: "By induction we will show that $P(n)$ is true for each integer $n \geq 1$.
(2) Base Case: Prove $P(1)$ is true.
(3) Inductive Hypothesis: Assume $P(k)$ is true for some arbitrary integer $k \geq 1$.
(4) Inductive Step: Using our inductive hypothesis, prove $P(k+1)$ is true.
(5) State our Conclusion: "Our result follows from induction."

1.

Find a formula for the following expression, where n is any positive integer:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

Use induction to prove that your formula is correct.

Solution:

(1) **Statement**: By induction we will show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \left(\frac{1}{2^i}\right) = 1 - \frac{1}{2^n}$$

(2) **Base Case**: n=1

$$\sum_{i=1}^{1} \left(\frac{1}{2^i}\right) = \frac{1}{2} = 1 - \frac{1}{2^1}$$

(3) **Inductive Hypothesis**: Assume that for some arbitrary integer $k \geq 1$,

$$\sum_{i=1}^{k} \left(\frac{1}{2^i}\right) = 1 - \frac{1}{2^k}$$

(4) **Inductive Step**: We will prove that the statement is true for $k+1$.

$$\sum_{i=1}^{k+1} \left(\frac{1}{2^i}\right) = \sum_{i=1}^{k} \left(\frac{1}{2^i}\right) + \frac{1}{2^{k+1}}$$

By our inductive hypothesis, we can substitute $1 - \frac{1}{2^k}$ for $\sum_{i=1}^{k} \left(\frac{1}{2^i}\right)$:

$$\sum_{i=1}^{k+1} \left(\frac{1}{2^i}\right) = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

(5) **Conclusion**: By induction, our statement is true for all integers $n \geq 1$.

2.

Use induction to prove that for all positive integers n:

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = \frac{(-1)^{n-1} n(n+1)}{2}$$

Solution:

(1) **Statement**: By induction we will show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \left((-1)^{i-1} i^2\right) = \frac{(-1)^{n-1} n(n+1)}{2}$$

(2) **Base Case**: n=1

$$\sum_{i=1}^{1} \left((-1)^{i-1} i^2\right) = (-1)^0 1^2 = 1 = \frac{(-1)^0 1(1+1)}{2}$$

(3) **Inductive Hypothesis**: Assume that for some arbitrary integer $k \geq 1$,

$$\sum_{i=1}^{k} \left((-1)^{i-1} i^2\right) = \frac{(-1)^{k-1} k(k+1)}{2}$$

(4) **Inductive Step**: We will prove that the statement is true for $k+1$.

$$\sum_{i=1}^{k+1} \left((-1)^{i-1} i^2\right) = \sum_{i=1}^{k} \left((-1)^{i-1} i^2\right) + (-1)^k (k+1)^2$$

By our inductive hypothesis, we can substitute $\frac{(-1)^{k-1}k(k+1)}{2}$ for $\sum_{i=1}^{k} \left((-1)^{i-1} i^2\right)$:

$$\begin{aligned}
\sum_{i=1}^{k+1} \left((-1)^{i-1} i^2\right) &= \frac{(-1)^{k-1} k(k+1)}{2} + (-1)^k (k+1)^2 \\
&= \frac{-1 * (-1)^k k(k+1)}{2} + (-1)^k (k+1)^2 \\
&= (-1)^k (k+1) \left(\frac{-k + 2(k+1)}{2}\right) \\
&= (-1)^k (k+1) \left(\frac{k+2}{2}\right) \\
&= \frac{(-1)^{(k+1)-1} (k+1)(k+2)}{2}
\end{aligned}$$

(5) **Conclusion**: By induction, our statement is true for all integers $n \geq 1$.

3.

Prove that for all positive integers n:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2$$

Hint: Try replacing the right hand side of the inequality with something that will make the statement stronger.

Solution: If we use the problem statement as our inductive hypothesis, then we run into trouble during our inductive step. For an arbitrary k, if all we know is that $\sum_{i=1}^{k} \left(\frac{1}{i^2}\right) \leq 2$, then it might actually be 2, in which case adding $\frac{1}{(k+1)^2}$ would make it too large. We will have the same problem no matter what constant we use; we must use an expression dependent on k. To prove the original statement, we need to first prove a stronger statement by induction.

(1) **Statement**: By induction we will show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \left(\frac{1}{i^2}\right) \leq 2 - \frac{1}{n}$$

(2) **Base Case**: n=1

$$\sum_{i=1}^{1}\left(\frac{1}{i^2}\right) = \frac{1}{1} \leq 2 - \frac{1}{1}$$

(3) **Inductive Hypothesis**: Assume that for some arbitrary integer $k \geq 1$,

$$\sum_{i=1}^{k}\left(\frac{1}{i^2}\right) \leq 2 - \frac{1}{k}$$

(4) **Inductive Step**: We will prove that the statement is true for $k+1$.

$$\sum_{i=1}^{k+1}\left(\frac{1}{i^2}\right) = \sum_{i=1}^{k}\left(\frac{1}{i^2}\right) + \frac{1}{(k+1)^2}$$

By our inductive hypothesis, we can substitute $2 - \frac{1}{k}$ for $\sum_{i=1}^{k}\left(\frac{1}{i^2}\right)$ and maintain the inequality:

$$\sum_{i=1}^{k+1}\left(\frac{1}{i^2}\right) \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)(k)} = 2 - \frac{k+1}{(k+1)(k)} + \frac{1}{(k+1)(k)}$$

$$= 2 - \frac{(k+1)-1}{(k+1)(k)} = 2 - \frac{k}{(k+1)(k)} = 2 - \frac{1}{(k+1)}$$

(5) **Conclusion**: By induction, for all integers $n \geq 1$,

$$\sum_{i=1}^{n}\left(\frac{1}{i^2}\right) \leq 2 - \frac{1}{n}$$

Finally, we will use the result we just proved to prove our original statement. For all integers $n \geq 1$, $\frac{1}{n} \geq 0$, so

$$\sum_{i=1}^{n}\left(\frac{1}{i^2}\right) \leq 2 - \frac{1}{n} \Rightarrow \sum_{i=1}^{n}\left(\frac{1}{i^2}\right) \leq 2$$

Challenge Problem: Horse Paradox

The following "proof" purports to show that all horses are the same color. Where is the error in the proof?

Statement: We will show that for any group of n horses, where n is a positive integer, all of them are the same color.
Base Case: $n = 1$. When there is only one horse in the "group", then clearly all horses in that group have the same color.
Inductive Hypotheis: Assume that for some arbitrary integer k, for any group of k horses, all of them are the same color.
Inductive step: We will prove that an arbitrary group of $k+1$ horses are all the same color. First, remove one horse. By our inductive hypothesis, the remaining group of k horses are all the same color. Next, add it back in and remove a different horse. Again, by our inductive hypothesis, the reamining group of k horses are all the same color. Since each horse we removed was the same color as the group when we removed the other one, all $k+1$ horses are the same color.
Conclusion: Any group of n horses are all the same color for any positive integer n. Therefore, all horses are the same color.

Solution: The error lies in our inductive step. If $k = 1$, then we try to prove the statement for $k = 2$. If we take an arbitrary group of 2 horses and remove one, the remaining "group" of 1 horse

is indeed a single color. However, if we remove the other horse, there are no horses in common between the two groups; therefore, we have no guarantee that all horses are the same color. When performing an inductive step, it is important that the step works for *all* $k \geq 1$, or else induction fails.
An additional explanation of the horse paradox can be found at
`https://en.wikipedia.org/wiki/All_horses_are_the_same_color`