Section Worksheet 7

Solutions

Structural Induction Format

How to prove $\forall x \in S.\ P(x)$ is true:

1. Let $P(x)$ be "…". We will prove $P(x)$ for all $x \in S$

2. Base Case: Show that P is true for all specific elements of S mentioned in the *Basis step*

3. Inductive Hypothesis: Assume that P is true for some arbitrary values of each of the existing named elements mentioned in the *Recursive step*

4. Inductive Step: Prove that P holds for each of the new elements constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

5. Conclude that $\forall x \in S.\ P(x)$

1. Regular Expressions

(a) The set of all strings of one or more 0s followed by a 1

Solution: 0*01

1. Regular Expressions

(b) The set of all strings of odd length

Solution: (0 ∪ 1)[(0 ∪ 1)(0 ∪ 1)]*

1. Regular Expressions

(c) **Caitlin's Sections:** The set of all strings not containing 000 and ending with a 1

Solution: (1 ∪ 01 ∪ 001)(1 ∪ 01 ∪ 001)*

(c) **Noah's Sections:** The set of all strings not containing 000

Solution: (((λ ∪ 0 ∪ 00)1)*) (λ ∪ 0 ∪ 00)

1. Regular Expressions

(d) The set of all strings containing a string of 1s such that the number of 1s is equivalent to 2 mod 3, followed by an even number of 0s

Solution: 11(111)*(00)*

2. Recursive Definitions and Structural Induction

Let S be the subset of the set of ordered pairs of integers defined recursively by:

Basis Step: (0,0) ∈ S

Recursive Step: If (a,b)∈S, then (a,b+1)∈S.
(a+1, b+1)∈S, and (a+2,b+1)∈S.

a) After first four applications of R.S.:

S = {(0,0),(0,1),(1,1),(2,1),(0,2)(1,2)(2,2),(3,2),
(4,2),(0,3),(1,3),(2,3),(3,3),(4,3),(5,3),(6,3),(0,4),
(1,4),(2,4),(3,4),(4,4),(5,4),(6,4),(7,4),(8,4)}

2. Recursive Definitions and Structural Induction

b) **Proof:**

1. Let P((a,b)) be "if (a,b)∈S, then a≤2b." Prove for all (a,b)∈S.

2. **Base Case:** (0,0)∈S by our basis step.

 0 ≤ 2(0) = 0, thus P((0,0)) is true.

3. **Inductive Hypothesis:** Assume P is true for some arbitrary values of each of the existing named elements mentioned in the recursive step. (i.e. P((a,b)) is true for arbitrary existing (a,b)∈S.)

4. **Inductive Step:** Must show that any new element generated by our recursive step makes P true.

i) Show P((a,b+1)) is true. **Goal:** Show a≤2(b+1).

      a≤2b by our I.H.

      Thus a≤2b+2=2(b+1), since 2b≤2b+2. ✓

ii) Show P((a+1,b+1)) is true. **Goal:** Show (a+1)≤2(b+1).

      a≤2b by our I.H.

      Thus a+1 ≤ 2b+1 (by adding 1 to both sides)

          ≤ 2b+2=2(b+1), since 2b+1≤2b+2. ✓

4. **Inductive Step:** Must show that any new element generated by our recursive step makes P true.

iii) Show P((a+2,b+1)) is true. **Goal:** Show (a+2)≤2(b+1).

   a≤2b by our I.H.

   Thus a+2 ≤ 2b+2 = 2(b+1) by adding 2 to both sides. ✓

5. **Conclusion:** We have shown our base case and inductive step, thus P((a,b)) is true for all (a,b)∈S by induction.

3. Context-Free Grammars

a) the set of all bit strings containing an even number of 0s and no 1s

Solution:

$$S \rightarrow \lambda \mid 00S$$

Explanation: Pretty self-explanatory; you keep generating two 0s at a time until you want to terminate, then you choose λ. This grammar also generates λ as a string; note that this is correct because zero is an even number and thus you can choose to generate zero 0s.

3. Context-Free Grammars

b) the set of all bit strings containing ten or more 0s and no 1s

Solution:

$$S \rightarrow 0000000000A$$
$$A \rightarrow 0A \mid \lambda$$

Explanation: We have to have at least ten 0s, thus we have our starting nonterminal add ten 0s and then jump to A, where you can choose to add more 0s (since we can have 'ten or more') or terminate.

3. Context-Free Grammars

c) the set of all bit strings containing an odd number of 0s followed by an even number of 1s

Solution:

$$S \to 0A$$

$$A \to 00A \mid 11B \mid \lambda$$

$$B \to 11B \mid \lambda$$

Explanation: We need at least one 0, thus we have our start symbol go to 0A only. There we can add on 2k 0s for some integer k to keep the number of 0s odd, start adding on an even number of 1s, or terminate (since zero is an even number, we don't need any 1s.)