Homework 3: Predicate Logic

CHANGELOG: This is version 4 (updated Monday 4:30 PM).
Missing parentheses in Problem 5 made the original stated version impossible. You may now either (1) complete the proof with the parentheses corrected or (2) demonstrate that our original claim was false (more details in the problem statement).
We corrected a typo in 7b: if the claim is **incorrect**, you should define $p, q, \ldots$.
We corrected a typo in the 'Theorem" line of problem 4 (initially we had $p \wedge q$, it should have been $p \wedge \neg q$).

Due date: Friday October 23 at 11:59 PM (Seattle time, i.e. GMT-7)
If you work with others (and you should!), remember to follow the collaboration policy.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting.

Be sure to read the grading guidelines for more information on what we're looking for.

This homework has 5 pages, make sure you keep scrolling!

1. Nested Quantifiers [15 points]

Fix your domain of discourse to be "all widgets"[1] (i.e. a single element of the domain is "a widget"). There are three types of widgets: red, blue, and yellow (every widget is exactly one of those types). The predicates $\texttt{red}(x), \texttt{blue}(x), \texttt{yellow}(x)$ return true if and only if the widget is of the named type. You can also use the predicates $\texttt{free}(x), \texttt{expensive}(x), \texttt{fancy}(x), \texttt{complicated}(x)$ to say a widget is free, expensive, fancy, or complicated respectively. Finally, use $\texttt{similar}(x, y)$ to say x and y are similar.

In this problem, an example of something you might give for a "scenario" might be "all fancy widgets are blue, but not all blue widgets are fancy"

(a) Your friend tried to translate "Every red widget is expensive or fancy" and got the following "$\forall x(\texttt{red}(x) \wedge \texttt{expensive}(x) \wedge \texttt{fancy}(x))$." The translation is incorrect. Give a correct translation, and describe a scenario (i.e. facts about widgets) in which your translation and their translation evaluate to different truth values.

(b) Your friend tried to translate "There is a blue widget that is similar to all yellow widgets" and got: "$\exists x \forall y([\texttt{blue}(x) \wedge \texttt{yellow}(y)] \rightarrow \texttt{similar}(x, y))$" The translation is incorrect. Give a correct translation, and describe a scenario (i.e. facts about widgets) in which your translation and their translation evaluate to different truth values.

(c) Translate the sentence "For every blue widget, there is a fancy widget such that for all red widgets: the blue widget and the fancy widget are similar, the red widget and blue widget are not similar, and the fancy widget is expensive" into predicate logic.

2. There is an implication [8 points]

Implications are uncommon under existential quantifiers. Consider this expression (which we'll call "the original expression"): $\exists x(P(x) \rightarrow Q(x))$

(a) Suppose that $P(x)$ is not always true (i.e. there is an element in the domain for which $P(x)$ is false). Explain why the original expression is true in this case. (1-2 sentences should suffice. If you prefer, you may give a formal proof instead).

[1] "widget" is an old-timey word for "something a machine would make" or "product." It's still occasionally used in CS as a more fancy sounding version of "thing."

(b) Suppose that $P(x)$ is always true (i.e. $\forall x P(x)$). There is a simpler statement which conveys the meaning of the original expression (i.e. is equivalent to it for all domains and predicates. By simpler, we mean "uses fewer symbols"). Give that expression, and briefly (1-2 sentences) explain why it works.

(c) Ponder, based on the last two parts, why it's very uncommon to write the original expression. You do not have to write anything for this part, simply ponder. [0 points]

3. For every iteration [6 points]

Imagine you have the predicate `pred`(x, y), which is true if and only if the java method `public boolean pred(Element x, Element y)` returns true. Write a java method that takes in a `Domain` object (which is a list of all the `Element`s in the domain) and returns the value of $\exists x \forall y$ `pred`(x, y)

You do not need to follow 142/143's style rules for code, but if your code is extremely unnecessarily convoluted you may lose points. We won't grade your code for java details (e.g. if you forget a semicolon, but it's clear what you meant we won't deduct; but errors that affect our understanding [say forgetting braces] may lead to deductions). You may want to consult Section 3's handout for examples of this type of code. If you're working in LaTeX you may want to use the verbatim environment (or just code in a text editor and insert a picture).

4. Spoof [14 points]

Theorem: Given $p \wedge \neg q$, $r \to s$, and $p \to \neg(\neg q \wedge s)$ prove $\neg r$.
"Spoof":

1.$p \wedge \neg q$	Given
2.p	$\wedge$ Elim: 1
3.$p \to \neg(\neg q \wedge s)$	Given
4.$\neg(\neg q \wedge s)$	MP: 2,3
5.$\neg\neg q \wedge \neg s$	DeMorgan's: 4
6.$q \wedge \neg s$	Double negation: 5
7.$\neg s$	$\wedge$ Elim: 6
8.$r \to s$	Given
9.$\neg s \to \neg r$	Contrapositive: 8
10.$\neg r$	MP: 7,9

(a) What is the most significant error in this proof? Give the line and briefly explain why it is wrong. [5 points]

(b) Show the theorem is true by fixing the error in the spoof. For this problem, please entirely rewrite the proof in your submission. [9 points]

5. Inference Proof [12 points]

Due to an error in the original statement, you may now do **either** part (a) or part (b) below.

(a) Using the logical inference rules and equivalences we have given, write an *inference proof* that given $\forall x([\exists y P(x, y)] \to \neg Q(x)), \forall x(\neg R(x) \to (Q(x) \vee \neg P(x, x)))$, and $\exists x\ P(x, x)$, you can conclude that $\exists x\ R(x)$.

You should consult the updated symbolic proof guidelines for our expectations on these proofs.

(b) The original version of the problem stated the first given as $\forall x(\exists y P(x, y) \to \neg Q(x))$ which (according to the rules from class) is equivalent to: $\forall x(\exists y[P(x, y) \to \neg Q(x)])$. With this given, the claim is **false**. Give predicates P, Q, R and a domain of discourse such that all of the givens (with the original version of the first given) are true, but the thing to prove is false. Briefly explain why your example works (1-2 sentences per given/conclusion should suffice).

6. Find The Bug [16 points]

The following proof claims to show that
Given: $\exists x P(x) \wedge \exists x Q(x)$, $\forall x(Q(x) \rightarrow R(x))$
Prove: $\exists x(P(x) \wedge R(x))$

1.$\exists x P(x) \wedge \exists x Q(x)$	Given
2.$P(c) \wedge Q(c)$	Eliminate $\exists$ (1)
3.$P(c)$	Eliminate $\wedge$ (2)
4.$Q(c)$	Eliminate $\wedge$ (2)
5.$\forall x(Q(x) \rightarrow R(x))$	Given
6.$Q(c) \rightarrow R(c)$	Eliminate $\forall$
7.$R(c)$	Modus Ponens (6,4)
8.$\exists x P(x)$	Introduce $\exists$
9.$\exists x R(x)$	Introdcue $\exists$
10.$\exists x(P(x) \wedge R(x))$	Introduce $\wedge$

(a) There is a bug in steps 1-4, where a rule is applied in a way that is not allowed. Identify the line where the rule is applied incorrectly, and explain why it is incorrect. [5 points]

(b) There is a bug in steps 5-10, **ignoring any mistakes that had happened before**, where a rule is applied incorrectly. Identify the line where this rule is applied incorrectly, and explain why it is incorrect. [5 points]

(c) Is the claim true?
If it is true, describe how to correct the proof. (You may say things like "replace step 3 with..." or "insert the following between steps 6 and 7..." or, if you prefer, you may rewrite the whole proof.
If the claim is false, describe P, Q, R and a domain of discourse such that the givens are true but the thing to prove is false. [6 points]

7. Inference Proof [20 points]

Theorem: Given $s \to (p \wedge q), \neg s \to r$, and $(r \vee p) \to q$, prove q.

"Spoof:"

1.	$\neg s \to r$	[Given]
2.	$(r \vee p) \to q$	[Given]
3.	$r \to q$	[Elim of $\vee$: 2]
	4.1. $\neg s$ [Assumption]	
	4.2. r [MP: 4.1, 1]	
	4.3. q [MP: 4.2, 3]	
4.	$\neg s \to q$	[Direct Proof Rule]
	5.1. s [Assumption]	
	5.2. $s \to (p \wedge q)$ [Given]	
	5.3. $p \wedge q$ [MP: 5.1, 5.2]	
	5.4. q [Elim of $\wedge$: 5.3]	
5.	$s \to q$	[Direct Proof Rule]
6.	$(s \to q) \wedge (\neg s \to q)$	[Intro $\wedge$: 5, 4]
7.	$(\neg s \vee q) \wedge (\neg\neg s \vee q)$	[Law of Implication]
8.	$(\neg s \vee q) \wedge (s \vee q)$	[Double Negation]
9.	$((\neg s \vee q) \wedge s) \vee ((\neg s \vee q) \wedge q)$	[Distributivity]
10.	$((\neg s \vee q) \wedge s) \vee (q \wedge (\neg s \vee q))$	[Commutativity]
11.	$((\neg s \vee q) \wedge s) \vee (q \wedge (q \vee \neg s))$	[Commutativity]
12.	$((\neg s \vee q) \wedge s) \vee q$	[Absorption]
13.	$(s \wedge (\neg s \vee q)) \vee q$	[Commutativity]
14.	$((s \wedge \neg s) \vee q) \vee q$	[Associativity]
15.	$(F \vee q) \vee q$	[Negation]
16.	$(q \vee F) \vee q$	[Commutativity]
17.	$q \vee q$	[Identity]
18.	q	[Idempotence]

(a) There are two major errors in this proof. Indicate which lines contain the errors and, for each one, explain (as briefly as possible) why that line is incorrect. [8 points]

(b) Is the conclusion of the "spoof" correct? If it is incorrect, describe propositions p, q, r, s such that the givens are true, but the claim is false. If the conclusion is correct, briefly explain how to correct any errors in lines 1–5 (you'll explain errors in 6–18 in part c). [4 points]

(c) Give a correct proof of what is claimed in lines 6–18, i.e., that from $(s \to q) \wedge (\neg s \to q)$, we can infer that q is true. [8 points]

8. Outline Labels Redux [4 points]

In this problem you'll label sections of formal proofs to give intuition for what is happening at a higher level than individual steps. For examples, you should look at the purple text in lecture 3 slide 31, or at what we describe as "intermediate goals" in HW2 problem 3a.

Because of confusion on HW1, 5c: of that part, and these two parts we will drop the worst single part for each student.

(a) Look at the canonical solution to Section 3 Number 6 (called "Formal Proof" in the handout). The proof has 14 steps. Label **portions** of the proof with high-level descriptions of what they are doing (instead of the names of the rule that make each individual step true). We divided this proof into 4 parts, but you might find a different division. Submit your answer in the form "Steps [X] to [Y]: [label]" for each part. [4 points]

(b) Do the same for the proof in Lecture 8 slide 20. Our division has 3 parts, but you might find a different division. [4 points]