Homework 4: Common Bugs and Comments

Since there were a few very common mistakes, we wanted to discuss what they were, and how to correct them.

1. Always introduce your variables.

Remember that even if a statement doesn't say "for all" it might implicitly be a for all (like the statement "even integers greater than 10 are not prime")

If you want to prove a for all, you need to introduce an arbitrary variable; for 311, you should always call arbitrary variables "arbitrary" (even if the statement we're proving doesn't call them arbitrary).

2. But only call a variable arbitrary if you mean it to be arbitrary.

If you're showing an existential, your variable is not arbitrary (at least it shouldn't be!).

Just because b is arbitrary does not mean that $b + 1$ is arbitrary (or any modification is arbitrary). Imagine you let b be an arbitrary natural number. $b + 1$ is not arbitrary – it's positive! $b + 1$ cannot be 0, so it's not arbitrary

You should use arbitrary to mean "I can put in any element here, and the rest of this argument will work." And furthermore, you should only use "arbitrary" if you intend to indicate that a for all statement can be derived at the end.

3. When proving an existential (or disproving a forall statement) give a very particular example.

Trying to give a general form (e.g. saying "choose any B, C where $B \subseteq C$ and you'll get a counter-example") is actually worse for your reader (and strictly speaking not a complete proof) even though it seems to give your reader more information.

You must give them a particular example; leaving it as "here's how to make an example" is incorrect – imagine you said "any prime number that is greater than 100 and even is a counter-example." There are no such numbers! Someone reading your proof has to verify not just your (now more abstract) proof of all your counter-examples, they also have to check the object you describe actually exists. Showing an object with the properties you claim exists is your job as someone proving an existential.

If your proof of an $\exists$ (or equivalently, disproof of a $\forall$) does not have a particular counter-example, it's not a full proof.

4. In problem 4a, we also saw a number of people saying essentially "an arbitrary element $b \in B$ is equal to an arbitrary eleemnt $c \in C$". This is not going to be correct – the claim you're asserting there is $\forall x \forall y([x \in B \wedge x \in C] \rightarrow b = c)$. That claim is not true, even if B and C are equal! (well unless $|B| \leq 1$ and $B = C$). You meant to show an arbitrary element of B is in C and an arbitrary element of C is in B. But that doesn't mean all elements are euqal.

Problem 4a, starting point, and done right

It was very common to start this proof with "Let (a, b) be an arbitrary element of $A \times B$. Let's break down the problem statement.

if $(A \times B = A \times C)$ then $B = C$.

So we will suppose:$(A \times B = A \times C)$ Our goal is to show $B = C$. That is $\forall x(x \in B \leftrightarrow x \in C)$.

So when you start a proof with "Let (a, b) be an arbitrary element of $A \times B$" you've gotten off to a wrong start. You need to show for an arbitrary x it's in B if and only if it's in C (or do both of those implications separately) – (a, b) isn't an element of either B or C.

Switching from an arbitrary (a, b) to an arbitrary b isn't particularly simple. You set yourself up to show $\forall a \forall b[(a, b) \in A \times B \rightarrow ???]$ and you need to end up at $\forall b(b \in B \leftrightarrow b \in C)$.

In a formal proof, what you'd have to do is introduce a particular a (this is possible, but tricky). But notice a key point here – you need to introduce a particular a – that a has to exist. With the start of an arbitrary (a, b) you're in danger of showing: $\forall a \forall b([a \in A \wedge b \in B] \leftrightarrow [a \in A \wedge b \in C])$.

That might not look that bad, but you can prove it in instances where $\forall b(b \in B \leftrightarrow b \in C)$ is false! If A is empty the statement $\forall a \forall b([a \in A \wedge b \in B] \leftrightarrow [a \in A \wedge b \in C])$. becomes vacuously true! But $\forall b(b \in B \leftarrow b \in C)$ (what you were actually trying to show) might not be.

So how does a correct proof go? You introduce an arbitrary $b \in B$, then **because** A **is nonempty** then you introduce a particular $a \in A$ (i.e. you say an $a \in A$ exists, rather than an arbitrary one). Since $a \in A$ and $b \in B$, $(a, b) \in A \times B$. Then the proof goes like most of yours did. The other direction (starting from an arbitrary $c \in C$) is essentially identical.

So how to avoid these errors?

(a) Make sure you understand exactly what the claim is asking you to show. If you aren't showing something is true $\forall a$, think carefully about whether you actually want/need that variable to be arbitrary.

(b) When you're done with a proof, read it through – have you told us whether each variable is arbitrary or just one that exists?

(c) When you read through your final proof, make sure that variables you introduced as arbitrary really are.