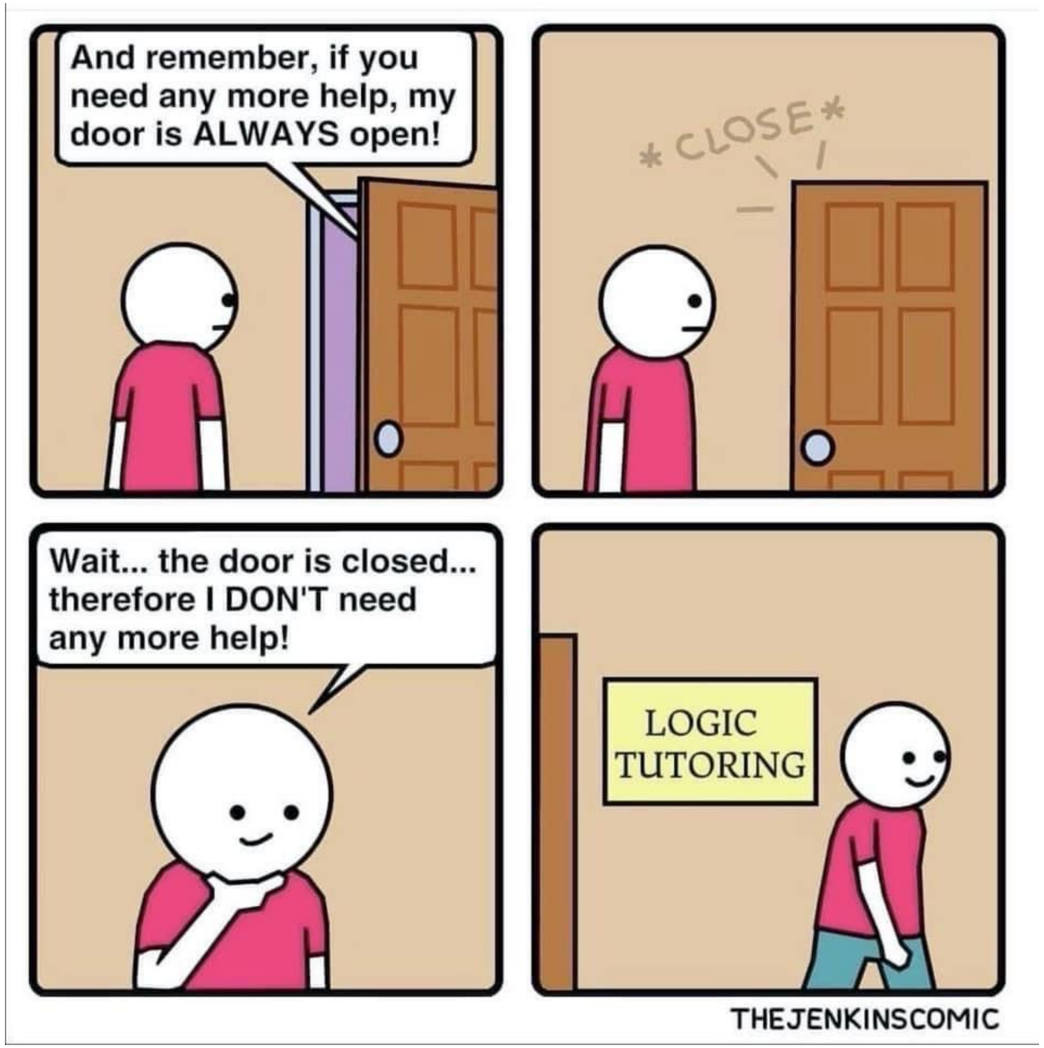

Warm up translate to predicate logic:
"For every x, if x is even, then $x = 2$."

Inference Proofs

Lecture 6

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

TRICK QUESTION! It depends on the domain.

Prime Numbers	Positive Integers	Odd integers
True	False	True (vacuously)

One Technical Matter

How do we parse sentences with quantifiers?

What's the "order of operations?"

We will usually put parentheses right after the quantifier and variable to make it clear what's included. If we don't, it's the full expression.

Parentheses often end up "not mattering" in real expressions.

Be careful with repeated variables…they don't always mean what you think they mean.

$\forall x\big(P(x)\big) \land \forall x(Q(x))$ are different x's.

Bound Variables

What happens if we repeat a variable?

Whenever you introduce a new quantifier with an already existing variable, it "takes over" that name until its expression ends.

$$\forall x(P(x) \land \forall x[Q(x)] \land R(x))$$

It's common (albeit somewhat confusing) practice to reuse a variables when it "wouldn't matter".

Never do something like the above: where a single name switches from gold to purple back to gold. Switching from gold to purple only is usually fine…but names are cheap.

More Practice

Let your domain of discourse be fruits.

There is a fruit that is tasty and ripe.

$$\exists x(\mathtt{Tasty}(x) \wedge \mathtt{Ripe}(x))$$

For every fruit, if it is not ripe then it is not tasty.

$$\forall x(\neg\mathtt{Ripe}(x) \rightarrow \neg\mathtt{Tasty}(x))$$

There is a fruit that is sliced and diced.

$$\exists x(\mathtt{Sliced}(x) \wedge \mathtt{Diced}(x))$$

This Week

This week we have two big topics:

Using and understanding quantifiers

Writing symbolic proofs (that aren't just simplifying)

Both of them are better if learned spaced out with practice, so…

…Every lecture this week is split in half, with a little bit on each topic.

Today: Tools for more complicated proofs.

Negating Quantifiers

Today

A new way of thinking of proofs:

Here's one way to get an iron-clad guarantee:

1. Write down all the facts we know.
2. Combine the things we know to derive new facts.
3. Continue until what we want to show is a fact.

Drawing Conclusions

You know "If it is raining, then I have my umbrella"

And "It is raining"

You should conclude…. I have my umbrella!

For whatever you conclude, convert the statement to propositional logic – will your statement hold for any propositions, or is it specific to raining and umbrellas?

I know $(p \rightarrow q)$ and p, I can conclude q
Or said another way: $[(p \rightarrow q) \wedge p] \rightarrow q$

Modus Ponens

The inference from the last slide is always valid. I.e.

$$[(p \rightarrow q) \land p] \rightarrow q \equiv \mathrm{T}$$

Modus Ponens – a formal proof

$[(p \to q) \land p] \to q$	$\equiv [(\neg p \lor q) \land p] \to q$	Law of Implication
	$\equiv [p \land (\neg p \lor q)] \to q$	Commutativity
	$\equiv [(p \land \neg p) \lor (p \land q)] \to q$	Distributivity
	$\equiv [\mathrm{F} \lor (p \land q)] \to q$	Negation
	$\equiv [(p \land q) \lor \mathrm{F}] \to q$	Commutativity
	$\equiv [(p \land q)] \to q$	Identity
	$\equiv [\neg(p \land q)] \lor q$	Law of Implication
	$\equiv [\neg p \lor \neg q] \lor q$	DeMorgan's Law
	$\equiv \neg p \lor [\neg q \lor q]$	Associativity
	$\equiv \neg p \lor [q \lor \neg q]$	Commutativity
	$\equiv \neg p \lor \mathrm{T}$	Negation
	$\equiv \mathrm{T}$	Domination

Modus Ponens

The inference from the last slide is always valid. I.e.

$$[(p \to q) \land p] \to q \equiv \mathrm{T}$$

We use that inference A LOT

So often people gave it a name ("Modus Ponens")

So often…we don't have time to repeat that 12 line proof EVERY TIME.

Let's make this another law we can apply in a single step.

Just like refactoring a method in code.

Notation – Laws of Inference

We're using the "$\rightarrow$ " symbol A LOT.

Too much

Some new notation to make our lives easier.

$$\frac{\text{If we know } \textbf{both } A \text{ and } B}{\therefore \quad \text{We can conclude any (or all) of } C, D} \qquad \frac{A, B}{\therefore \quad C, D}$$

"$\therefore$" means "therefore" – I knew A, B therefore I can conclude C, D.

$$\frac{p \rightarrow q, p}{\therefore \quad q}$$

Modus Ponens, i.e. $[(p \rightarrow q) \land p] \rightarrow q)$, in our new notation.

Another Proof

Let’s keep going.

I know “If it is raining then I have my umbrella” and “I do not have my umbrella”

I can conclude… It is not raining!

What’s the general form? $[(p \to q) \wedge \neg q] \to \neg p$

How do you think the proof will go?

If you had to convince a friend of this claim in English, how would you do it?

A proof!

We know $p \to q$ and $\neg q$; we want to conclude $\neg p$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

A proof!

We know $p \to q$ and $\neg q$; we want to conclude $\neg p$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

1. $p \to q$	Given
2. $\neg q$	Given
3. $\neg q \to \neg p$	Contrapositive of 1.
4. $\neg p$	Modus Ponens on 3,2.

Try it yourselves

Suppose you know $p \rightarrow q$, $\neg s \rightarrow \neg q$,and p.
Give an argument to conclude s.

Fill out the poll everywhere for Activity Credit!

Go to pollev.com/cse311 and login with your UW identity
Or text cse311 to 22333

Try it yourselves

Suppose you know $p \rightarrow q, \neg s \rightarrow \neg q$,and p.
Give an argument to conclude s.

1.	$p \rightarrow q$	Given
2.	$\neg s \rightarrow \neg q$	Given
3.	p	Given
4.	q	Modus Ponens 1,3
5.	$q \rightarrow s$	Contrapositive of 2
6.	s	Modus Ponens 5,4

More Inference Rules

We need a couple more inference rules.

These rules set us up to get facts in exactly the right form to apply the really useful rules.

A lot like commutativity and distributivity in the propositional logic rules.

Eliminate ∧	$$\frac{A \wedge B}{\therefore \quad A, B}$$	$$\frac{\text{I know the fact } A \wedge B}{\therefore \text{ I can conclude } A \text{ is a fact and } B \text{ is a fact } \textbf{separately}.}$$

More Inference Rules

In total, we have two for ∧ and two for ∨, one to create the connector, and one to remove it.

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \wedge B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

None of these rules are surprising, but they are useful.

The Direct Proof Rule

We've been implicitly using another "rule" today, the direct proof rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \rightarrow B}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…starting next time.

Caution

Be careful! Logical inference rules can only be applied to **entire** facts. They cannot be applied to portions of a statement (the way our propositional rules could). Why not?

Suppose we know $p \rightarrow q, r$. Can we conclude q?

1.	$p \rightarrow q$	Given
2.	r	Given
3.	$(p \vee r) \rightarrow q$	Introduce ∨ (1)
4.	$p \vee r$	Introduce ∨ (2)
5.	q	Modus Ponens 3,4.

Intro ∨ $\dfrac{A}{\therefore A \vee B, B \vee A}$

One more Proof

Show if we know: $p, q, [(p \land q) \rightarrow (r \land s)], r \rightarrow t$ we can conclude t.

One more Proof

Show if we know: $p, q, [(p \land q) \to (r \land s)], r \to t$ we can conclude t.

1.	p	Given
2.	q	Given
3.	$[(p \land q) \to (r \land s)]$	Given
4.	$r \to t$	Given
5.	$p \land q$	Intro $\land$ (1,2)
6.	$r \land s$	Modus Ponens (3,5)
7.	r	Eliminate $\land$ (6)
8.	t	Modus Ponens (4,7)

Inference Rules

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{P \rightarrow Q; P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

Quantifiers

Quantifiers

∀ (for **A**ll) and ∃ (there **E**xists)

Write these statements in predicate logic with quantifiers. Let your domain of discourse be “cats”

This sentence implicitly makes a statement about all cats!

If a cat is fat, then it is happy.

$$\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$$

Quantifiers

Writing implications can be tricky when we change the domain of discourse.

If a cat is fat, then it is happy.

Domain of Discourse: cats $\forall x[\mathtt{Fat}(x) \rightarrow \mathtt{Happy}(x)]$

What if we change our domain of discourse to be all mammals?
We need to limit x to be a cat. How do we do that?

$\forall x[(\mathtt{Cat}(x) \wedge \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)]$ $\forall x[\mathtt{Cat}(x) \wedge (\mathtt{Fat}(x) \rightarrow \mathtt{Happy}(x))]$

Quantifiers

Which of these translates "If a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\texttt{Cat}(x) \wedge \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$

For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\texttt{Cat}(x) \wedge (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[what if x is a dog? Dogs are in the domain, but…uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

Quantifiers

Existential quantifiers need a different rule:

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

There is a dog who is not happy.

Domain of discourse: dogs
$\exists x(\neg\, \texttt{Happy}(x))$

Quantifiers

Which of these translates "There is a dog who is not happy."
when our domain of discourse is "mammals"?

$\exists x[\texttt{Dog}(x) \rightarrow \neg\texttt{Happy}(x))]$

There is a mammal, such that if x is a dog then it is not happy.
[this can't be right – plug in a cat for x and the implication is true]

$\exists x[(\texttt{Dog}(x) \land \neg\texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[this one is correct!]

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

Negating Quantifiers

What happens when we negate an expression with quantifiers?

What does your intuition say?

Original	Negation
Every positive integer is prime	There is a positive integer that is not prime.
$\forall x\ \texttt{Prime}(x)$ Domain of discourse: positive integers	$\exists x(\neg\ \texttt{Prime}(x))$ Domain of discourse: positive integers

Negating Quantifiers

Let's try on an existential quantifier…

Original	Negation
There is a positive integer which is prime and even.	Every positive integer is composite or odd.
$\exists x(\texttt{Prime}(x) \wedge \texttt{Even}(x))$ Domain of discourse: positive integers	$\forall x(\neg\texttt{Prime}(x) \vee \neg\texttt{Even}(x))$ Domain of discourse: positive integers

To negate an expression with a quantifier

1. Switch the quantifier ($\forall$ becomes $\exists$, $\exists$ becomes $\forall$)
2. Negate the expression inside

Negation

Translate these sentences to predicate logic, then negate them.

All cats have nine lives.

$$\forall x\big(Cat(x) \rightarrow NumLives(x, 9)\big)$$

$\exists x(Cat(x) \wedge \neg\big(NumLives(x, 9)\big))$ "There is a cat without 9 lives.

All dogs love every person.

$$\forall x \forall y\big(Dog(x) \wedge Human(y) \rightarrow Love(x, y)\big)$$

$\exists x \exists y(Dog(x) \wedge Human(y) \wedge \neg Love(x, y))$ "There is a dog who does not love someone." "There is a dog and a person such that the dog doesn't love that person."

There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \wedge Human(y) \wedge Love(x, y)$$

$$\forall x \forall y(Cat(x) \wedge Human(y) \rightarrow \neg Love(x, y))$$

"For every cat and every human, the cat does not love that human."

"Every cat does not love any human" ("no cat loves any human")

Negation with Domain Restriction

$\exists x \exists y (Cat(x) \land Human(y) \land Love(x, y))$

$\forall x \forall y (Cat(x) \land Human(y) \rightarrow \neg Love(x, y))$

There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?

Next Time

For every cat, there is a human that it loves.

Translating sentences with both kinds of quantifiers.