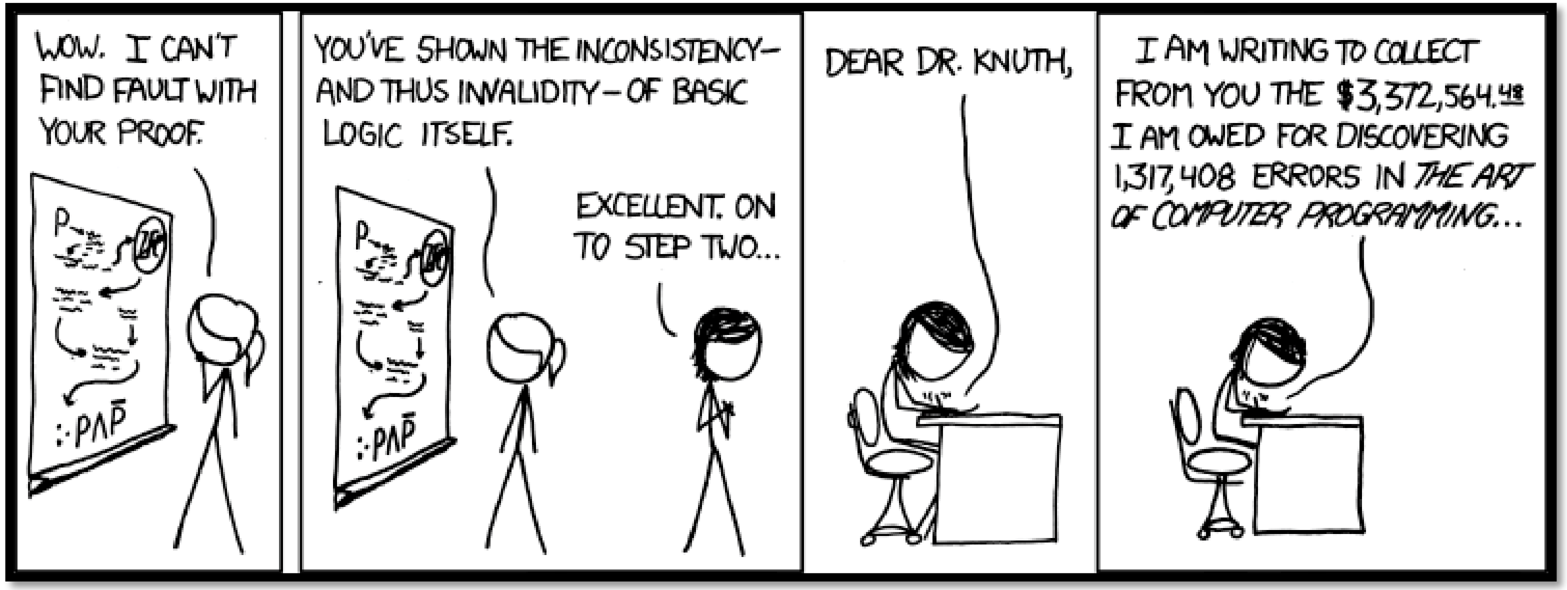

xkcd.com/816/

Inference Proofs With Quantifiers

Announcements

HW1 came back yesterday.

Do take a look today, so you don't repeat mistakes from HW1 to HW2.

HW1 5c (the label the proof with your intuition part) did not go as I planned.

About 15% of the class interpreted that part as saying "label the individual step with rule names"

1. This was the first time a 311 course has asked for this kind of thing – we didn't find clear wording; that's on me.
2. We did model the type of question in lecture, and got questions on Ed clarifying what was meant. – I think there were enough resources that everyone should have been able to understand.

Announcements

About 15% of you didn’t even try the problem (because you didn’t think there was anything to do)

That means you didn’t learn. Which is the opposite of what I want.

HW3 has two more “give us a summary” questions. (doing “5c” again on different proofs). Of the three parts, we’ll drop the lowest score.

More resources on domain restriction coming soon!

Given: $((p \to q) \wedge (q \to r))$
Show: $(p \to r)$

Here's a corrected version of the proof.

1. $(p \to q) \wedge (q \to r)$ Given
2. $p \to q$ Eliminate ∧ 1
3. $q \to r$ Eliminate ∧ 1
   - 4.1 p Assumption
   - 4.2 q Modus Ponens 4.1,2
   - 4.3 r Modus Ponens 4.2,3
5. $p \to r$ Direct Proof Rule

When introducing an assumption to prove an implication: Indent, and change numbering.

When reached your conclusion, use the Direct Proof Rule to observe the implication is a fact.

The conclusion is an unconditional fact (doesn't depend on p) so it goes back up a level

Try it!

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

Rule	Premises	Conclusion
Eliminate ∧	$A \wedge B$	$\therefore A, B$
Eliminate ∨	$A \vee B, \neg A$	$\therefore B$
Intro ∧	$A; B$	$\therefore A \wedge B$
Intro ∨	A	$\therefore A \vee B, B \vee A$
Direct Proof rule	$A \Rightarrow B$	$A \rightarrow B$
Modus Ponens	$P \rightarrow Q; P$	$\therefore Q$

You can still use all the propositional logic equivalences too!

Inference Rules

Eliminate ∧

$$\frac{A \land B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \lor B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \land B}$$

Intro ∨

$$\frac{A}{\therefore A \lor B, B \lor A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{P \rightarrow Q; P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Excluded Middle

$$\frac{}{\therefore \quad A \lor \neg A}$$

DeMorgan's (Quantifiers)

$$\neg(\forall x\, A) \equiv \exists x(\neg A)$$
$$\neg(\exists x A) \equiv \forall x(\neg A)$$

Try it!

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

1. $p \vee q$ Given
2. $(r \wedge s) \rightarrow \neg q$ Given
3. r Given
    - 4.1 s Assumption
    - 4.2 $r \wedge s$ Intro $\wedge$ (3,4.1)
    - 4.3 $\neg q$ Modus Ponens (2, 4.2)
    - 4.4 $q \vee p$ Commutativity (1)
    - 4.5 p Eliminate $\vee$ (4.4, 4.3)
5. $s \rightarrow p$ Direct Proof Rule

Try it!

Given: $p \lor q$, $(r \land s) \to \neg q$, r.
Show: $s \to p$

1. $p \lor q$ Given
2. $(r \land s) \to \neg q$ Given
3. r Given
   - 4.1 s Assumption
   - 4.2 $r \land s$ Intro $\land$ (3,4.1)
   - 4.3 $\neg q$ Modus Ponens (2, 4.2)
   - 4.4 $q \lor p$ Commutativity (1)
   - 4.5 p Eliminate $\lor$ (4.4, 4.3)
5. $s \to p$ Direct Proof Rule

Introduce givens

assume s and set up to conclude $\neg q$

use $\neg q$ to conclude p

Finish with DPR

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

Let's see a good example, then come back to those "arbitrary" and "fresh" conditions.

Proof Using Quantifiers

Suppose we know $\exists x P(x)$ and $\forall y[\ P(y) \rightarrow Q(y)]$. Conclude $\exists x Q(x)$.

Eliminate ∀

$$\frac{\forall x\ P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\ P(x)}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore \quad \forall x\ P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \ P(c) \text{ for a fresh } c}$$

Proof Using Quantifiers

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \rightarrow Q(y)]$. Conclude $\exists x Q(x)$.

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Proof Using Quantifiers

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \rightarrow Q(y)]$. Conclude $\exists x Q(x)$.

1. $\exists x P(x)$ Given
2. $P(a)$ Eliminate ∃ 1
3. $\forall y[P(y) \rightarrow Q(y)]$ Given
4. $P(a) \rightarrow Q(a)$ Eliminate ∀ 3
5. $Q(a)$ Modus Ponens 2,4
6. $\exists x Q(x)$ Intro ∃ 5

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for a fresh } c}$$

"arbitrary" means a is "just" a variable in our domain.
It doesn't depend on any other variables and wasn't introduced with other information.

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a); a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for a fresh } c}$$

"fresh" means c is a new symbol (there isn't another c somewhere else in our proof).

Fresh and Arbitrary

Suppose we know $\exists x P(x)$. Can we conclude $\forall x P(x)$?

1. $\exists x\ P(x)$ Given
2. $P(a)$ Eliminate $\exists$ (1)
3. $\forall x\ P(x)$ Intro $\forall$ (2)

This proof is **definitely** wrong.
(take $P(x)$ to be "is a prime number")

a wasn't **arbitrary**. We knew something about it – it's the x that exists to make $P(x)$ true.

Intro $\exists$

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\ P(x)}$$

Eliminate $\exists$

$$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$$

Eliminate $\forall$

$$\frac{\forall x\ P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro $\forall$

$$\frac{P(a);\ a \text{ is arbitrary}}{\therefore\ \forall x\ P(x)}$$

Fresh and Arbitrary

Intro ∀

$$\frac{P(a);\ a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

You can trust a variable to be **arbitrary** if you introduce it as such.

If you eliminated a ∀ to create a variable, that variable is arbitrary. Otherwise it's not arbitrary – it depends on something.

You can trust a variable to be **fresh** if the variable doesn't appear anywhere else (i.e. just use a new letter)

Fresh and Arbitrary

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

There are no similar concerns with these two rules.

Want to reuse a variable when you eliminate ∀? Go ahead.

Have a c that depends on many other variables, and want to intro ∃?

Also not a problem.

Arbitrary

In section yesterday, you said: $[\exists y \forall x\ P(x, y)] \rightarrow [\forall x \exists y\ P(x, y)]$. Let's prove it!!

Arbitrary

In section yesterday, you said: $[\exists y \forall x\ P(x,y)] \to [\forall x \exists y\ P(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\ P(x,y)$	Assumption
1.2 $\forall x\ P(x,c)$	Elim $\exists$ (1.1)
1.3 Let a be arbitrary.	--
1.4 $P(a,c)$	Elim $\forall$ (1.2)
1.5 $\exists y\ P(a,y)$	Intro $\exists$ (1.4)
1.6 $\forall x \exists y\ P(x,y)$	Intro $\forall$ (1.5)
2. $[\exists y \forall x\ P(x,y)] \to [\forall x \exists y\ P(x,y)]$	Direct Proof Rule

Arbitrary

In section yesterday, you said: $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\, P(x,y)$ Assumption
1.2 $\forall x\, P(x,c)$ Elim $\exists$ (1.1)

1.4 $P(a,c)$ Elim $\forall$ (1.2)
1.5 $\exists y\, P(a,y)$ Intro $\exists$ (1.4)
1.6 $\forall x \exists y\, P(x,y)$ Intro $\forall$ (1.5)

2. $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$ Direct Proof Rule

Find The Bug

Let your domain of discourse be integers.
We claim that given $\forall x \exists y$ Greater(y, x), we can conclude $\exists y \forall x$ Greater(y, x)
Where Greater(y, x) means $y > x$

1.	$\forall x \exists y$ Greater(y, x)	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y$ Greater(y, a)	Elim $\forall$ (1)
4.	$b \geq a$	Elim $\exists$ (2)
5.	$\forall x$ Greater(b, x)	Intro $\forall$ (4)
6.	$\exists y \forall x$ Greater(y, x)	Intro $\exists$ (5)

Find The Bug

1. $\forall x \exists y$ `Greater`(y, x) Given
2. Let a be an arbitrary integer --
3. $\exists y$ `Greater`(y, a) Elim ∀ (1)
4. $b \geq a$ Elim ∃ (2)
5. $\forall x$ `Greater`(b, x) Intro ∀ (4)
6. $\exists y \forall x$ `Greater`(y, x) Intro ∃ (5)

b is not arbitrary. The variable b depends on a. Even though a is arbitrary, b is not!

Bug Found

There's one other "hidden" requirement to introduce $\forall$.

"No other variable in the statement can depend on the variable to be generalized"

Think of it like this -- b was probably $a + 1$ in that example.

You wouldn't have generalized from `Greater`$(a + 1, a)$

To $\forall x$ `Greater`$(a + 1, x)$. There's still an a, you'd have replaced all the a's.

x depends on y if y is in a statement when x is introduced.

This issue is much clearer in English proofs, which we'll start next time.