Homework 4: English proofs

Due date: Wednesday April 28 at 11:00 PM (Seattle time, i.e. GMT-7)
If you work with others (and you should!), remember to follow the collaboration policy.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting.

Be sure to read the grading guidelines for more information on what we're looking for.

This is the first homework with English proofs. Please look again at the grading guidelines now that we're doing English proofs.

While working on the assignment, please (roughly) keep track of the time you spent on each problem. This is so that you have an idea of how to answer the feedback questions!

1. Collaborators

List any collaborators (i.e., other students) you worked with and which problems you worked on together, or state that you worked alone.

2. English Spoofs and Confusions [18 points]

2.1. Lions and tigers and bears (oh my!) [6 points]

A grizzly bear stole a toddler from a pair of parents. The grizzly bear promises that he will return the child if and only if the parents correctly predict whether or not the grizzly will return the child.

(a) Imagine the parents predict that the child will be returned. In this case, will the bear always, sometimes, or never keep his promise? Show your reasoning. *Hint: Consider the two choices that the bear can make and whether or not these choices are consistent with his promise.*

(b) Imagine the parents predict that the child will **not** be returned. In this case, will the bear always, sometimes, or never keep his promise? Show your reasoning.

(c) The grizzly then says "if I am unable to keep my original promise, then I will return the child." Assume that the bear is telling the truth. What prediction should the parents make? Show your reasoning.

2.2. Sugar makes everything better [6 points]

Assume the following things to be true.

Ice cream is delicious, and dessert this evening is something delicious. The desserts which were available at the store today were ice cream and cake. Dessert this evening was purchased at the store today.

Consider the following (incorrect) proof of the proposition "Tonight's dessert must be ice cream."

Tonight's dessert must be ice cream because:

(a) tonight's dessert was purchased at the store today (by assumption), (b) and ice cream and cake were the only two desserts available at the store today (by assumption), and (c) we know that tonight's dessert is delicious, (d) and ice cream is delicious (by assumption), (e) so we conclude that tonight's dessert is ice cream.

(a) Identify the most significant error in the proof and discuss why this step is incorrect. Sentences have been labeled to easily refer back to specific portions of the proof.

(b) If the given statement is true, write a correct proof. If it is false, provide a counterexample.

2.3. What's wrong with this picture? [6 points]

Consider the following statement: for all real numbers a and b, if $a^2 = b^2$ then $a = b$.

And the following spoof (incorrect proof) of the statement:

> Let a and b be real numbers such that $a^2 = b^2$. Since $a^2 \geq 0, b^2 \geq 0$, their square root is a real number and nonnegative. Then, applying the square root function to both sides, we conclude $a = \sqrt{a^2} = \sqrt{b^2} = b$.

(a) Why is the above proof incorrect?

(b) Is the original statement true or false? If the statement is true, write a correct proof. If it is false, provide a counterexample.

3. Formal and English [20 points]

In this problem, we'll practice writing both Formal and English proofs. Let your domain of discourse be integers.
Define $\texttt{Even}(x)$ to be true if and only if $\exists k(x = 2k)$.
Define $\texttt{divBy6}(x)$ to be true if and only if $6 \mid x$.

(a) Give a predicate definition of $\texttt{divBy6}(x)$, that uses an $\exists$ quantifier. [2 points]

(b) Show that $\forall x(\texttt{divBy6}(x) \to \texttt{Even}(x))$, using an inference proof. Let the domain of discourse for your proof be integers.
You may use the definitions of predicates in the problem (including your answer to part a), as well as "algebra" to complete the proof. [8 points]

(c) Write an English proof to show the if 6 divides an integer x, then x is even. Recall that English proofs don't have domains of discourse, so you need to define types for your variables. [8 points]

(d) Go through your English proof, for each sentence in it, state which step(s) of your inference proof it most closely corresponds to (it's ok if a few steps overlap or don't correspond to a particular sentence, but this shouldn't happen to a lot of steps.). [2 points]

4. English Proofs (6 points)

Provide an English proof for the following statement: *For any integers x, y, and z, if $x + y$ is even and $y + z$ is even, then also $x + z$ is even.*

5. Set Proofs [16 points]

Let A, B, C be arbitrary sets. For each of the following claims: if it is true, give an **English proof**. If it is false, disprove it with an English proof (If you need to disprove the statement, remember that we've seen only one proof technique in class for disproving a $\forall$).

(a) $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$.

(b) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$

6. I've never seen such raw power[sets] [16 points]

Let S, T be arbitrary sets. For each of the following claims: if it is true, give an English proof. If it is false, disprove it with an English proof (if you need to disprove the statement, remember that we've seen only one proof technique in class for disproving a $\forall$).

(a) $\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T) \cup \mathcal{P}(S \cap T)$

(b) $\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)$.

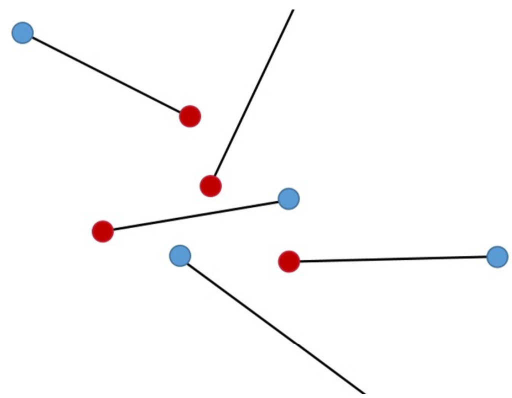
Note: This problem is challenging. We recommend starting in advance, once we've introduced the power set definition in Friday's lecture.

7. Cartesian Products [15 points]

Let A, B, and C be arbitrary sets. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

8. Extra Credit: Matchmaker, Matchmaker, Make Me a Match

In this problem, you will show that given n red points and n blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are n red and n blue points fixed in the plane.



A *matching* M is a collection of n line segments connecting distinct red-blue pairs. The *total length* of a matching M is the sum of the lengths of the line segments in M. Say that a matching M is *minimal* if there is no matching with a smaller total length.

Let $\mathsf{IsMinimal}(M)$ be the predicate that is true precisely when M is a minimal matching. Let $\mathsf{HasCrossing}(M)$ be the predicate that is true precisely when there are two line segments in M that cross each other.

Give an argument in English explaining why there must be at least one matching M so that $\mathsf{IsMinimal}(M)$ is true, i.e.

$$\exists M \mathsf{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M(\mathsf{HasCrossing}(M) \to \neg\mathsf{IsMinimal}(M))$$

Now use the two results above to give a proof of the statement:

$$\exists M \neg\mathsf{HasCrossing}(M).$$

Note: length is defined as usual (in Euclidean space).

9. Feedback

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment?
- Which problem did you spend the most time on?
- Which problem did you find to be the most confusing?
- Any other feedback for us?