CSE 311: Foundations of Computing

Lecture 20: Regular expressions

Recap: Structural Induction

Consider a recursively defined set S:

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.

Recap: Structural Induction

Consider a recursively defined set S:

- *Basis step:* Some specific elements are in S
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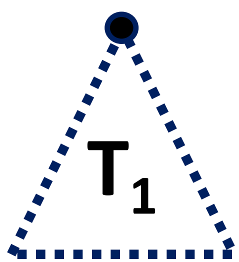
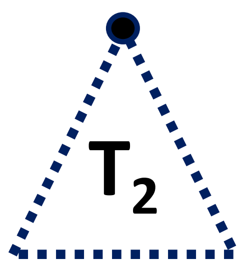
How to prove $\forall\, x \in S, P(x)$ is true:

- Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*
- Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*
- Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis
- Conclude that $\forall\, x \in S, P(x)$

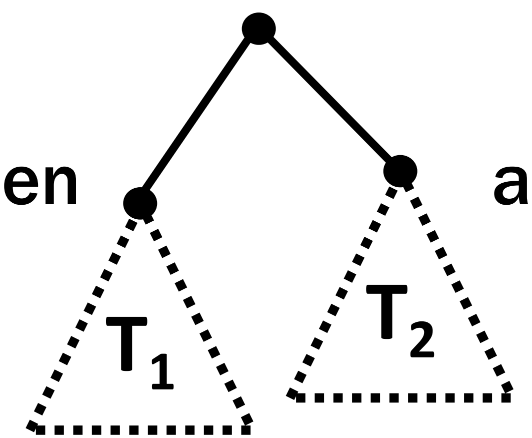
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

and 

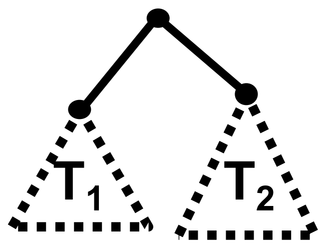
are rooted binary trees,

then 

also is a rooted binary tree.

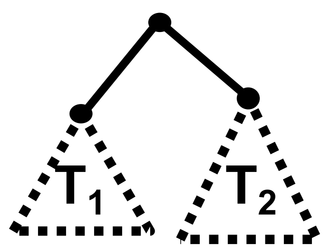
Defining Functions on Rooted Binary Trees

- size(•) = 1

- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)

- height(•) = 0

- height (

)=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Claim: **For every rooted binary tree T,** size(**T**) $\leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

Claim: For every rooted binary tree **T**, $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T})+1} - 1$

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1. Let $P(\mathbf{T})$ be "$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$". We prove $P(\mathbf{T})$ for all rooted binary trees **T** by structural induction.

Claim: **For every rooted binary tree T,** size(**T**) $\leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

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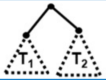
1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height(T)}+1}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0, **and** $2^{0+1}-1=2^1-1=1$ **so** P(•) **is true.**

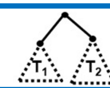
Claim: **For every rooted binary tree T,** size(**T**) $\leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

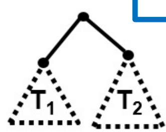
1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height(T)+1}}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0, **and** $2^{0+1}-1=2^1-1=1$ **so** P(•) **is true.**
3. **Inductive Hypothesis: Suppose that** P($\mathbf{T_1}$) **and** P($\mathbf{T_2}$) **are true for some rooted binary trees** $\mathbf{T_1}$ **and** $\mathbf{T_2}$.
4. **Inductive Step:**

Goal: Prove P().

Claim: **For every rooted binary tree T,** size(**T**) $\leq 2^{\text{height}(\mathbf{T})\,+\,1} - 1$

1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height}(T)+1} - 1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0, **and** $2^{0+1}-1=2^1-1=1$ **so** P(•) **is true.**
3. **Inductive Hypothesis: Suppose that** P(**T**$_1$) **and** P(**T**$_2$) **are true for some rooted binary trees T**$_1$ **and T**$_2$**.**
4. **Inductive Step:**

Goal: Prove P().

By defn, size() $= 1 + \text{size}(\mathbf{T_1}) + \text{size}(\mathbf{T_2})$

$\leq 1 + 2^{\text{height}(\mathbf{T_1})+1} - 1 + 2^{\text{height}(\mathbf{T_2})+1} - 1$

by IH for **T**$_1$ and **T**$_2$

$\leq 2^{\text{height}(\mathbf{T_1})+1} + 2^{\text{height}(\mathbf{T_2})+1} - 1$

$\leq 2(2^{\max(\text{height}(\mathbf{T_1}),\text{height}(\mathbf{T_2}))+1}) - 1$

$\leq 2(2^{\text{height}($  $)}) - 1 \leq 2^{\text{height}($  $)+1} - 1$

which is what we wanted to show.

5. **So, the** P(T) **is true for all rooted bin. trees by structural induction.**

Languages: Sets of Strings

- Sets of strings that satisfy special properties are called *languages*. Examples:
  - English sentences
  - Syntactically correct Java/C/C++ programs
  - $\Sigma^* =$ All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Legal variable names. keywords in Java/C/C++
  - Binary strings with an equal # of 0's and 1's

Regular Expressions

Regular expressions over Σ

- **Basis**:
  - ε is a regular expression
  - $\varnothing$ is a regular expression
  - a is a regular expression for any $a \in \Sigma$
- **Recursive step**:
  - If **A** and **B** are regular expressions then so are:
    - $\mathbf{A} \cup \mathbf{B}$
    - $\mathbf{AB}$
    - $\mathbf{A^*}$

Each Regular Expression is a "pattern"

ε matches the **empty string**

$\varnothing$ does not match any string

a matches the one character string *a*

A $\cup$ **B** matches all strings that either **A** matches or **B** matches (or both)

AB matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another

Examples

*001**

Examples

001*

{00, 001, 0011, 00111, ...}

Examples

$(0 \cup 1)\, 0\, (0 \cup 1)\, 0$

$(0^*1^*)^*$

Examples

$(\mathit{0} \cup \mathit{1})\ \mathit{0}\ (\mathit{0} \cup \mathit{1})\ \mathit{0}$

{0000, 0010, 1000, 1010}

$(\mathit{0}^*\mathit{1}^*)^*$

All binary strings

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

Binary strings that contain "0110"

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Binary strings that begin with pairs of characters followed by "01010" or "10001"

Lecture 20 Activity

- You will be assigned to **breakout rooms**. Please:
- Introduce yourself
- Choose someone to share screen, showing this PDF

Which of these is the language of 0^*1^*?

a) The set of all binary strings with any 0s coming before all 1s
b) The set of all binary strings starting starting with 0 and ending with 1
c) The set of all binary strings
d) The set of all binary strings where every 0 is followed by a 1.

Fill out a poll everywhere for Activity Credit!
Go to pollev.com/thomas311 and login
with your UW identity

Regular Expressions in Practice

- Used to define the "tokens": e.g., legal variable names, keywords in programming languages and compilers
- Legal variable names in Java are

$$(a \cup \cdots \cup z \cup A \cup \cdots \cup Z \cup \$ \cup \_)(a \cup \cdots \cup z \cup A \cup \cdots \cup Z \cup \$ \cup \_ \cup 1 \cup \cdots \cup 9)^*$$

- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of PHP
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

- Pattern p = Pattern.compile("a*b");
- Matcher m = p.matcher("aaaaab");
- boolean b = m.matches();

  `[01]` a 0 or a 1 `^` start of string `$` end of string

  `[0-9]` any single digit `\.` period `\,` comma `\-` minus

  `.` any single character

ab	a followed by b	(**AB**)
(a|b)	a or b	(**A** $\cup$ **B**)
a?	zero or one of a	(**A** $\cup$ ε)
a*	zero or more of a	**A***
a+	one or more of a	**AA***

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)

Limitations of Regular Expressions

- Not all languages can be specified by regular expressions
- Even some easy things like
  - Palindromes
  - Strings with equal number of 0’s and 1’s
- But also more complicated structures in programming languages
  - Matched parentheses
  - Properly formed arithmetic expressions
  - etc.