Section 03: Solutions

1. Domain Restriction

Translate each of the following sentences into logical notation. These translations require some of our quantifier tricks. You may use the operators $+$ and $\cdot$ which take two numbers as input and evaluate to their sum or product, respectively. Remember:

- To restrict the domain under a $\forall$ quantifier, add a hypothesis to an implication.
- To restrict the domain under an $\exists$ quantifier, AND in the restriction.
- If you want variables to be different, you have to explicitly require them to be not equal.

(a) Domain: Positive integers; Predicates: $\texttt{Even}, \texttt{Prime}, \texttt{Equal}$
"There is only one positive integer that is prime and even." **Solution:**

$\exists x(\texttt{Prime}(x) \land \texttt{Even}(x) \land \forall y[\neg \texttt{Equal}(x, y) \to \neg(\texttt{Even}(y) \land \texttt{Prime}(y))])$

(b) Domain: Real numbers; Predicates: $\texttt{Even}, \texttt{Prime}, \texttt{Equal}$
"There are two different prime numbers that sum to an even number." **Solution:**

$\exists x \exists y(\texttt{Prime}(x) \land \texttt{Prime}(y) \land \neg Equal(x, y) \land \texttt{Even}(x + y))$

(c) Domain: Real numbers; Predicates: $\texttt{Even}, \texttt{Prime}, \texttt{Equal}$
"The product of two distinct prime numbers is not prime." **Solution:**

$\forall x \forall y([\texttt{Prime}(x) \land \texttt{Prime}(y) \land \neg \texttt{Equal}(x, y)] \to \neg \texttt{Prime}(xy))$

(d) Domain: Real numbers; Predicates: $\texttt{Even}, \texttt{Prime}, \texttt{Equal}, \texttt{Postivite}, \texttt{Greater}, \texttt{Integer}$
"For every positive integer, there is a greater even integer" **Solution:**

$\forall x(\texttt{Positive}(x) \land \texttt{Integer}(x) \to [\exists y(\texttt{Integer}(y) \land \texttt{Even}(y) \land \texttt{Greater}(y, x))])$
Or equivalently: $\forall x \exists y(\texttt{Positive}(x) \land \texttt{Integer}(x) \to (\texttt{Integer}(y) \land \texttt{Even}(y) \land \texttt{Greater}(y, x)))$

2. Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

(a) $\forall x\ \forall y\ P(x, y)$ $\qquad$ $\forall y\ \forall x\ P(x, y)$ **Solution:**

These sentences are the same; switching universal quantifiers makes no difference.

(b) $\exists x\ \exists y\ P(x, y)$ $\qquad$ $\exists y\ \exists x\ P(x, y)$ **Solution:**

These sentences are the same; switching existential quantifiers makes no difference.

(c) $\forall x\ \exists y\ P(x,y)$ $\qquad$ $\forall y\ \exists x\ P(x,y)$ **Solution:**

These are only the same if P is symmetric (i.e., the order of the arguments doesn't matter). If the order of the arguments does matter, then these are different statements. For instance, if $P(x,y)$ is "$x < y$", then the first statement says "for every x, there is a corresponding y such that $x < y$", whereas the second says "for every y, there is a corresponding x such that $x < y$". In other words, in the first statement y is a function of x, and in the second x is a function of y.

If your domain of discourse is "positive integers", for example, the first is true and the second is false; but for "negative integers" the second is true while the first is false.

(d) $\forall x\ \exists y\ P(x,y)$ $\qquad$ $\exists x\ \forall y\ P(x,y)$ **Solution:**

These two statements are usually different.

(e) $\forall x\ \exists y\ P(x,y)$ $\qquad$ $\exists y\ \forall x\ P(x,y)$ **Solution:**

The second statement is "stronger" than the first (that is, the second implies the first). For the first, y is allowed to depend on x. For the second, one specific y must work for all x. Thus if the second is true, whatever value of y makes it true, can also be plugged in for y in the first statement for every x. On the other hand, if the first statement is true, it might be that different y's work for the different x's and no single value of y exists to make the latter true.

As an example, let you domain of discourse be positive real numbers, and let $P(x,y)$ be $xy = 1$. The first statement is true (always take y to be $1/x$, which is another positive real number). The second statement is not true; it asks for a single number that always makes the product 1.

3. Quantifier Ordering

Let your domain of discourse be a set of `Element` objects given in a list called `Domain`. Imagine you have a predicate `pred`(x,y), which is encoded in the java method `public boolean pred(int x, int y)`. That is you call your predicate `pred` true if and only if the java method returns `true`.

(a) Consider the following Java method:

```
public boolean Mystery(Domain D){
    for(Element x : D) {
        for(Element y : D) {
            if(pred(x,y))
                return true;
        }
    }
}
```

`Mystery` corresponds to a quantified formula (for `D` being the domain of discourse), what is that formula?
Solution:

$\exists x \exists y($`pred`$(x,y))$. If any combination of x and y causes pred to evaluate to true, we return true; that is we just want x, y to exist.

(b) What formula does `mystery2` correspond to

```
public boolean Mystery2(Domain D){
```

```
    for(Element x : D) {
        boolean thisXPass = false;
        for(Element y : D) {
            if(pred(x,y))
                thisXPass = true;
        }
        if(!thisXPass)
            return false;
    }
    return true;
}
```

Solution:

$\forall x \exists y(\texttt{pred}(x, y))$.

For a given x, when we come across a y that makes $\texttt{pred}(x, y)$ true, we set the given x to pass (so one y suffices for a given x) but we require **every** x to pass, so x is universally quantified. Since y is allowed to depend on x, we have x as the outermost variable.

4. Find the Bug

Each of these inference proofs is incorrect. Identify the line (or lines) which incorrectly apply a law, and explain the error. Then, if the claim is false, give concrete examples of propositions to show it is false. If it is true, write a correct proof.

(a) This proof claims to show that given $a \to (b \vee c)$, we can conclude $a \to c$.

1. $a \to (b \vee c)$ [Given]
   - 2.1. a [Assumption]
   - 2.2. $\neg b$ [Assumption]
   - 2.3. $b \vee c$ [Modus Ponens, from 1 and 2.1]
   - 2.4. c [$\vee$ elimination, from 2.2 and 2.3]
2. $a \to c$ [Direct Proof Rule, from 2.1-2.4]

Solution:

The error here is in lines 2.2 and 2. When beginning a subproof for the direct proof rule, only one assumption may be introduced. If the author of this proof wanted to assume both a and $\neg b$, they should have used the assumption $a \wedge \neg b$, which would make line 3 be $(a \wedge \neg b) \to c$ instead.

And the claim is false in general. Consider:
a: "I am outside"
b: "I am walking my dog"
c: "I am swimming"
If we assert "If I am outside, I am walking my dog or swimming," we cannot reasonably conclude that "If I am outside, I am swimming" ($a \to c$).

(b) This proof claims to show that given $p \to q$ and r, we can conclude $p \to (q \vee r)$.

1.$p \to q$ [Given]
2.r [Given]
3.$p \to (q \vee r)$ [Intro $\vee$ (1,2)]

Solution:

Bug is in step 3, we're applying the rule to only a subexpression.

The statement is true though. A correct proof introduces p as an assumption, uses MP to get q, introduces $\vee$ to get $q \vee r$, and the direct proof rule to complete the argument.

(c) This proof claims to show that given $p \to q$ and q that we can conclude p

$1. p \to q$	[Given]
$2. q$	[Given]
$3. \neg p \vee q$	[Law of Implication (1)]
$4. q$	[Eliminate $\vee$ (2,3)

Solution:

The bug is in step 4. Eliminate $\vee$ from 3 would let us conclude $\neg p$ if we had $\neg q$ or q if we had p. It doesn't tell us anything with knowing q.

Indeed, the claim is false. We could have p: "it is raining"
q: "I have my umbrella"

And be a person who always carries their umbella with them (even on sunny days). The information is not sufficient to conclude p.

5. Formal Spoofs

For each of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $\exists z\, \forall x\, P(x, z)$ follows from $\forall x\, \exists y\, P(x, y)$.

1.	$\forall x\, \exists y\, P(x, y)$	[Given]
2.	$\forall x\, P(x, c)$	[$\exists$ Elim: 1, c special]
3.	$\exists z\, \forall x P(x, z)$	[$\exists$ Intro: 2]

Solution:

The mistake is on line 2 where an inference rule is used on a subexpression. When we apply something like the $\exists$ Elim rule, the $\exists$ must be at the start of the expression and outside all other parts of the statement.

The conclusion is false, it's basically saying we can interchange the order of $\forall$ and $\exists$ quantifiers. Let the domain of discourse be integers and define $P(x, y)$ to be $x < y$. Then the hypothesis is true: for every integer, there is a larger integer. However, the conclusion is false: there is no integer that is larger than every other integer. Hence, there can be no correct proof that the conclusion follows from the hypothesis.

(b) Show that $\exists z\, (P(z) \wedge Q(z))$ follows from $\forall x\, P(x)$ and $\exists y\, Q(y)$.

1.	$\forall x\, P(x)$	[Given]
2.	$\exists y\, Q(y)$	[Given]
3.	Let z be arbitrary	
4.	$P(z)$	[$\forall$ Elim: 1]
5.	$Q(z)$	[$\exists$ Elim: 2, let z be the object that satisfies $Q(z)$]
6.	$P(z) \wedge Q(z)$	[$\wedge$ Intro: 4, 5]
7.	$\exists z\, P(z) \wedge Q(z)$	[$\exists$ Intro: 6]

Solution:

The mistake is on line 5. The $\exists$ Elim rule must create a new variable rather than applying some property to an existing variable.

The conclusion is true in this case. Instead of declaring z to be arbitrary and then applying $\exists$ Elim to make it specific, we can instead just apply the $\exists$ Elim rule directly to create z. To do this, we would remove lines 3 and 5 and define z by applying $\exists$ Elim to line 2. Note, it's important that we define z before applying line 4.

6. Formal Proof (Direct Proof Rule)

Show that $\neg t \to s$ follows from $t \vee q$, $q \to r$ and $r \to s$. **Solution:**

1.	$t \vee q$	[Given]
2.	$q \to r$	[Given]
3.	$r \to s$	[Given]
4.1.	$\neg t$	[Assumption]
4.2.	q	[Elim of $\vee$: 1, 4.1]
4.3.	r	[MP of 4.2, 2]
4.4.	s	[MP 4.3, 3]
4.	$\neg t \to s$	[Direct Proof Rule]

7. Predicate Logic Formal Proof

Given $\forall x.\, T(x) \to M(x)$, we wish to prove $(\exists x.\, T(x)) \to (\exists y.\, M(y))$. The following formal proof does this, but it is missing citations for which rules are used, and which lines they are based on. Fill in the blanks with inference rules or predicate logic equivalences, as well as the line numbers.

Then, summarize in English what is going on here.

1. $\forall x.\, T(x) \to M(x)$ (________________)
   - 2.1. $\exists x.\, T(x)$ (________________)
   - Let r be the object that satisfies $T(r)$
   - 2.2. $T(r)$ (________________, from ______)
   - 2.3. $T(r) \to M(r)$ (________________, from ______)
   - 2.4. $M(r)$ (________________, from ______)
   - 2.5. $\exists y.\, M(y)$ (________________, from ______)
2. $(\exists x.\, T(x)) \to (\exists y.\, M(y))$ (________________, from ______)

Solution:

1. $\forall x.\, T(x) \to M(x)$ (Given)
   - 2.1. $\exists x.\, T(x)$ (Assumption)
   - Let r be the object that satisfies $T(r)$
   - 2.2. $T(r)$ ($\exists$ elimination, from 2.1)
   - 2.3. $T(r) \to M(r)$ ($\forall$ elimination, from 1)
   - 2.4. $M(r)$ (Modus Ponens, from 2.2 and 2.3)
   - 2.5. $\exists y.\, M(y)$ ($\exists$ introduction, from 2.4)

2. $(\exists x. T(x)) \to (\exists y. M(y))$ (Direct Proof Rule, from 2.1-2.5)

Following the premise of the implication, we suppose there is an object that satisfies $T(\cdot)$. Then it must satisfy $M(\cdot)$ also, by the given, which gives us the conclusion of the implication.

8. Formal Proof

Show that $\neg p$ follows from $\neg(\neg r \vee t)$, $\neg q \vee \neg s$ and $(p \to q) \wedge (r \to s)$. **Solution:**

1.	$\neg(\neg r \vee t)$	[Given]
2.	$\neg q \vee \neg s$	[Given]
3.	$(p \to q) \wedge (r \to s)$	[Given]
4.	$\neg\neg r \wedge \neg t$	[DeMorgan's Law: 1]
5.	$\neg\neg r$	[Elim of $\wedge$: 4]
6.	r	[Double Negation: 5]
7.	$r \to s$	[Elim of $\wedge$: 3]
8.	s	[MP, 6,7]
9.	$\neg\neg s$	[Double Negation: 8]
10.	$\neg s \vee \neg q$	[Commutative: 2]
11.	$\neg q$	[Elim of $\vee$: 10, 9]
12.	$p \to q$	[Elim of $\wedge$: 3]
13.	$\neg q \to \neg p$	[Contrapositive: 12]
14.	$\neg p$	[MP: 11,13]

9. A Formal Proof in Predicate Logic

Prove $\exists x\ (P(x) \vee R(x))$ from $\forall x\ (P(x) \vee Q(x))$ and $\forall y\ (\neg Q(y) \vee R(y))$. **Solution:**

1.	$\forall x\ (P(x) \vee Q(x))$	[Given]
2.	$\forall y\ (\neg Q(y) \vee R(y))$	[Given]
3.	$P(a) \vee Q(a)$	[Elim $\forall$: 1]
4.	$\neg Q(a) \vee R(a)$	[Elim $\forall$: 2]
5.	$Q(a) \to R(a)$	[Law of Implication: 4]
6.	$\neg\neg P(a) \vee Q(a)$	[Double Negation: 3]
7.	$\neg P(a) \to Q(a)$	[Law of Implication: 5]
	8.1. $\neg P(a)$ [Assumption]	
	8.2. $Q(a)$ [Modus Ponens: 8.1, 7]	
	8.3. $R(a)$ [Modus Ponens: 8.2, 5]	
8.	$\neg P(a) \to R(a)$	[Direct Proof]
9.	$\neg\neg P(a) \vee R(a)$	[Law of Implication: 8]
10.	$P(a) \vee R(a)$	[Double Negation: 9]
11.	$\exists x\ (P(x) \vee R(x))$	[Intro $\exists$: 10]