

# Even More Induction

CSE 311 Autumn 20  
Lecture 17

# Monochromatic Cows

Consider the following statements.

Suppose each cow is only one color.

Every group of cows contain only cows of the same color.

Spoof by strong induction:

$P(1)$ : all groups of cows of size one contain only cows of one color.

(IH):  $P(k) \rightarrow P(k + 1)$  for all  $k \geq 1$ .

Consider a group of  $k + 1 \geq 2$  cows.

By our IH, all groups of size  $\geq k$  contain cows of only one color.

So, break the  $k + 1$  cows into 2 overlapping subgroups;

these by our IH are all the same color,

and since they share a cow in common the 2 groups are the same color.

Fill out the poll everywhere for  
Activity Credit!

Go to [pollev.com/cse311](https://pollev.com/cse311) and  
login with your UW identity  
Or text cse311 to 22333

# A good last check

After you've finished writing an inductive proof, pause.

If your inductive step always goes back  $s$  steps, you need  $s$  base cases (otherwise  $b + 1$  will go back before the base cases you've shown). And make sure your inductive hypothesis is strong enough.

If your inductive step is going back a varying (unknown) number of steps, check the first few values above the base case, make sure your cases are really covered. And make sure your IH is strong.

# Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define  $P(n)$ . State that your proof is by induction on  $n$ .
2. Base Cases: Show  $P(b_{min}), P(b_{min+1}) \dots P(b_{max})$  i.e. show the base cases
3. Inductive Hypothesis: Suppose  $P(b_{min}) \wedge P(b_{min} + 1) \wedge \dots \wedge P(k)$  for an arbitrary  $k \geq b_{max}$ . (The smallest value of  $k$  assumes all bases cases, but nothing else)
4. Inductive Step: Show  $P(k + 1)$  (i.e. get  $[P(b_{min}) \wedge \dots \wedge P(k)] \rightarrow P(k + 1)$ )
5. Conclude by saying  $P(n)$  is true for all  $n \geq b_{min}$  by the principle of induction.

# Strong Induction vs. Weak Induction

Think of strong induction as “my recursive call might be on LOTS of smaller values” (like mergesort – you cut your array in half)

Think of weak induction as “my recursive call is always on one step smaller.”

Practical advice:

A strong hypothesis isn't wrong when you only need a weak one (but a weak one is wrong when you need a strong one). Some people just always write strong hypotheses. But it's easier to typo a strong hypothesis.

Robbie leaves a blank spot where the IH is, and fills it in after the step.



# Stamp Collection (attempt)

Define  $P(n)$  I can make  $n$  cents of stamps with just 4 and 5 cent stamps.

We prove  $P(n)$  is true for all  $n \geq 12$  by induction on  $n$ .

Base Case:

12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose [maybe some other stuff and]  $P(k)$ , for an arbitrary  $k \geq 12$ .

Inductive Step:

We want to make  $k + 1$  cents of stamps. By IH we can make  $k - 3$  cents exactly with stamps. Adding another 4 cent stamp gives exactly  $k + 1$  cents.

# Stamp Collection

Is the proof right?

How do we know  $P(13)$

We're not the base case, so our inductive hypothesis assumes  $P(12)$ , and then we say if  $P(9)$  then  $P(13)$ .

Wait a second....

If you go back  $s$  steps every time, you need  $s$  base cases.

Or else the first few values aren't proven.



# Stamp Collection

Define  $P(n)$  I can make  $n$  cents of stamps with just 4 and 5 cent stamps.

We prove  $P(n)$  is true for all  $n \geq 12$  by induction on  $n$ .

Base Case:

12 cents can be made with three 4 cent stamps.

13 cents can be made with two 4 cent stamps and one 5 cent stamp.

14 cents can be made with one 4 cent stamp and two 5 cent stamps.

15 cents can be made with three 5 cent stamps.

Inductive Hypothesis Suppose  $P(12) \wedge P(13) \wedge \dots \wedge P(k)$ , for an arbitrary  $k \geq 15$ .

Inductive Step:

We want to make  $k + 1$  cents of stamps. By IH we can make  $k - 3$  cents exactly with stamps. Adding another 4 cent stamp gives exactly  $k + 1$  cents.

# A good last check

After you've finished writing an inductive proof, pause.

If your inductive step always goes back  $s$  steps, you need  $s$  base cases (otherwise  $b + 1$  will go back before the base cases you've shown). And make sure your inductive hypothesis is strong enough.

If your inductive step is going back a varying (unknown) number of steps, check the first few values above the base case, make sure your cases are really covered. And make sure your IH is strong.

# Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define  $P(n)$ . State that your proof is by induction on  $n$ .
2. Base Cases: Show  $P(b_{min}), P(b_{min+1}) \dots P(b_{max})$  i.e. show the base cases
3. Inductive Hypothesis: Suppose  $P(b_{min}) \wedge P(b_{min} + 1) \wedge \dots \wedge P(k)$  for an arbitrary  $k \geq b_{max}$ . (The smallest value of  $k$  assumes all bases cases, but nothing else)
4. Inductive Step: Show  $P(k + 1)$  (i.e. get  $[P(b_{min}) \wedge \dots \wedge P(k)] \rightarrow P(k + 1)$ )
5. Conclude by saying  $P(n)$  is true for all  $n \geq b_{min}$  by the principle of induction.

# Practical Advice

How many base cases do you need?

Always at least one.

If you're analyzing recursive code or a recursive function, at least one for each base case of the code/function.

If you always go back  $s$  steps, at least  $s$  consecutive base cases.

Enough to make sure every case is handled.

# Stamp Collection, Done Wrong

Define  $P(n)$  I can make  $n$  cents of stamps with just 4 and 5 cent stamps.

We prove  $P(n)$  is true for all  $n \geq 12$  by induction on  $n$ .

Base Case:

12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose  $P(k)$ ,  $k \geq 12$ .

Inductive Step:

We want to make  $k + 1$  cents of stamps. By IH we can make  $k$  cents exactly with stamps. Replace one of the 4 cent stamps with a 5 cent stamp.

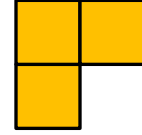
$P(n)$  holds for all  $n$  by the principle of induction.

# Stamp Collection, Done Wrong

What if the starting point doesn't have any 4 cent stamps?

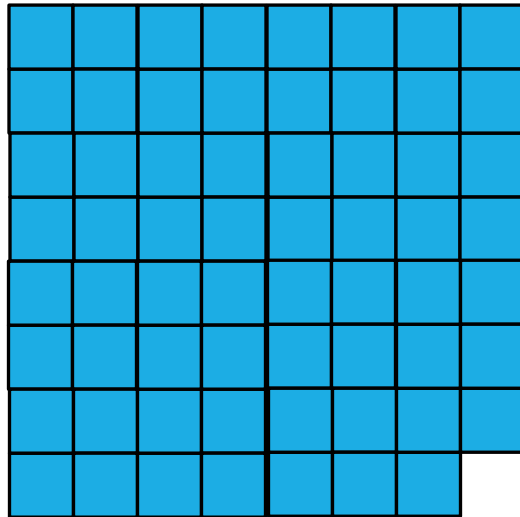
Like, say, 15 cents =  $5+5+5$ .

# Gridding



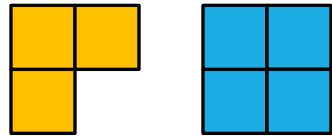
I've got a bunch of these 3 piece tiles.

I want to fill a  $2^n \times 2^n$  grid ( $n \geq 1$ ) with the pieces, except for a  $1 \times 1$  spot in a corner.



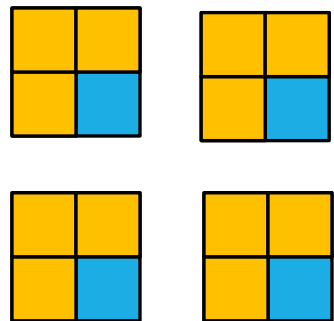
# Gridding: Not a formal proof, just a sketch

Base Case:  $n = 1$



Inductive hypothesis: Suppose you can tile a  $2^k \times 2^k$  grid, except for a corner.

Inductive step:  $2^{k+1} \times 2^{k+1}$ , divide into quarters. By IH can tile...





# Recursively Defined Functions

Just like induction works well with recursive code, it also works well for recursively-defined functions.

Define the Fibonacci numbers as follows:

$$f(0) = 1$$

$$f(1) = 1$$

$$f(n) = f(n - 1) + f(n - 2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

\*This is a somewhat unusual definition,  $f(0) = 0, f(1) = 1$  is more common.

# Fibonacci Inequality

Show that  $f(n) \leq 2^n$  for all  $n \geq 0$  by induction.

$$\begin{aligned} f(0) &= 1; & f(1) &= 1 \\ f(n) &= f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2. \end{aligned}$$

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

# Fibonacci Inequality

Show that  $f(n) \leq 2^n$  for all  $n \geq 0$  by induction.

Define  $P(n)$  to be " $f(n) \leq 2^n$ ". We show  $P(n)$  is true for all  $n \geq 0$  by induction on  $n$ .

Base Cases: ( $n = 0$ ):  $f(0) = 1 \leq 1 = 2^0$ .

( $n = 1$ ):  $f(1) = 1 \leq 2 = 2^1$ .

Inductive Hypothesis: Suppose  $P(0) \wedge P(1) \wedge \dots \wedge P(k)$  for an arbitrary  $k \geq 1$ .

Inductive step:

Target:  $P(k+1)$ . i.e.  $f(k+1) \leq 2^{k+1}$

# Fibonacci Inequality

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that  $f(n) \leq 2^n$  for all  $n \geq 0$  by induction.

Define  $P(n)$  to be " $f(n) \leq 2^n$ ". We show  $P(n)$  is true for all  $n \geq 0$  by induction on  $n$ .

Base Cases: ( $n = 0$ ):  $f(0) = 1 \leq 1 = 2^0$ .

( $n = 1$ ):  $f(1) = 1 \leq 2 = 2^1$ .

Inductive Hypothesis: Suppose  $P(0) \wedge P(1) \wedge \dots \wedge P(k)$  for an arbitrary  $k \geq 1$ .

Inductive step:  $f(k+1) = f(k) + f(k-1)$  by the definition of the Fibonacci numbers. Applying IH twice, we have  $f(k+1) \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$ .

Therefore, we have  $P(n)$  for all  $n \geq 0$  by the principle of induction.

Claim:  $3 \mid (2^{2n} - 1)$  for all  $n \in \mathbb{N}$ .

[Define  $P(n)$ ]

Base Case

Inductive Hypothesis

Inductive Step

[conclusion]

Claim:  $3 \mid (2^{2^n} - 1)$  for all  $n \in \mathbb{N}$ .

Let  $P(n)$  be " $3 \mid (2^{2^n} - 1)$ ." We show  $P(n)$  holds for all  $n \in \mathbb{N}$ .

Base Case ( $n = 0$ ) note that  $2^{2^n} - 1 = 2^0 - 1 = 0$ . Since  $3 \cdot 0 = 0$ , and 0 is an integer,  $3 \mid (2^{2 \cdot 0} - 1)$ .

Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary  $k \geq 0$

Inductive Step:

Target:  $P(k + 1)$ , i.e.  $3 \mid (2^{2^{(k+1)}} - 1)$

Therefore, we have  $P(n)$  for all  $n \in \mathbb{N}$  by the principle of induction.

**Claim:**  $3 \mid (2^{2n} - 1)$  for all  $n \in \mathbb{N}$ .

Let  $P(n)$  be " $3 \mid (2^{2n} - 1)$ ." We show  $P(n)$  holds for all  $n \in \mathbb{N}$ .

Base Case ( $n = 0$ ) note that  $2^{2n} - 1 = 2^0 - 1 = 0$ . Since  $3 \cdot 0 = 0$ , and 0 is an integer,  $3 \mid (2^{2 \cdot 0} - 1)$ .

Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary  $k \geq 0$

Inductive Step: By inductive hypothesis,  $3 \mid (2^{2k} - 1)$ . i.e. there is an integer  $j$  such that  $3j = 2^{2k} - 1$ .

$$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$$

**FORCE the expression in your IH to appear**

Target:  $P(k + 1)$ , i.e.  $3 \mid (2^{2(k+1)} - 1)$

Therefore, we have  $P(n)$  for all  $n \in \mathbb{N}$  by the principle of induction.

**Claim:**  $3 \mid (2^{2n} - 1)$  for all  $n \in \mathbb{N}$ .

Let  $P(n)$  be " $3 \mid (2^{2n} - 1)$ ." We show  $P(n)$  holds for all  $n \in \mathbb{N}$ .

Base Case ( $n = 0$ ) note that  $2^{2n} - 1 = 2^0 - 1 = 0$ . Since  $3 \cdot 0 = 0$ , and 0 is an integer,  $3 \mid (2^{2 \cdot 0} - 1)$ .

Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary  $k \geq 0$

Inductive Step: By inductive hypothesis,  $3 \mid (2^{2k} - 1)$ . i.e. there is an integer  $j$  such that  $3j = 2^{2k} - 1$ .

$$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4(2^{2k} - 1) + 4 - 1$$

By IH, we can replace  $2^{2k} - 1$  with  $3j$  for an integer  $j$

$$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$$

Since  $4j + 1$  is an integer, we meet the definition of divides and we have:

Target:  $P(k + 1)$ , i.e.  $3 \mid (2^{2(k+1)} - 1)$

Therefore, we have  $P(n)$  for all  $n \in \mathbb{N}$  by the principle of induction.



Claim:  $3 \mid (2^{2^n} - 1)$  for all  $n \in \mathbb{N}$ .

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$$2^{2 \cdot 0} - 1 = 0 = 3 \cdot 0$$

$$2^{2 \cdot 1} - 1 = 3 = 3 \cdot 1$$

$$2^{2 \cdot 2} - 1 = 15 = 3 \cdot 5$$

$$2^{2 \cdot 3} - 1 = 63 = 3 \cdot 21$$

$$2^{2 \cdot 4} - 1 = 255 = 3 \cdot 85$$

$$2^{2 \cdot 5} - 1 = 1023 = 3 \cdot 341$$

The divisor goes from  $k$  to  $4k + 1$

$$0 \rightarrow 4 \cdot 0 + 1 = 1$$

$$1 \rightarrow 4 \cdot 1 + 1 = 5$$

$$5 \rightarrow 4 \cdot 5 + 1 = 21$$

...

That might give us a hint that  $4k + 1$  will be in the algebra somewhere, and give us another intermediate target.

# Induction: Hats!

You have  $n$  people in a line ( $n \geq 2$ ). Each of them wears either a **purple hat** or a **gold hat**. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that there is a person with a purple hat next to someone with a gold hat.

Yes this is kinda obvious. I promise this is good induction practice.

Yes you could argue this by contradiction. I promise this is good induction practice.

# Induction: Hats!

Define  $P(n)$  to be "in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show  $P(n)$  for all integers  $n \geq 2$  by induction on  $n$ .

Base Case:  $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have  $P(n)$  for all  $n \geq 2$

# Induction: Hats!

Define  $P(n)$  to be "in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show  $P(n)$  for all integers  $n \geq 2$  by induction on  $n$ .

Base Case:  $n = 2$  The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary  $k \geq 2$ .

Inductive Step: Consider a line with  $k + 1$  people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have  $P(n)$  for all  $n \geq 2$

# Induction: Hats!

Define  $P(n)$  to be "in a line of gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show  $P(n)$  for all integers  $n \geq 2$  by induction on  $n$ .

Base Case:  $n = 2$  The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary  $k \geq 2$ .

Inductive Step: Consider a line with  $k + 1$  people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2: There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length  $k$ , has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have  $P(k + 1)$ .

By the principle of induction, we have  $P(n)$  for all  $n \geq 2$

# Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that  $f(n) \geq 2^{n/2}$  for all  $n \geq 2$  by induction.

[Define  $P(n)$ ]

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have  $P(n)$  for all  $n \geq 0$  by the principle of induction.

# Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that  $f(n) \geq 2^{n/2}$  for all  $n \geq 2$  by induction.

Define  $P(n)$  to be " $f(n) \geq 2^{n/2}$ " We show  $P(n)$  is true for all  $n \geq 2$  by induction on  $n$ .

Base Cases:  $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$$

Inductive Hypothesis: Suppose  $P(2) \wedge P(3) \wedge \dots \wedge P(k)$  for an arbitrary  $k \geq 3$ .

Inductive step:  $f(k+1) = f(k) + f(k-1)$  by the definition of the Fibonacci numbers. Applying IH twice, we have

Target:  $f(k+1) \geq 2^{(k+1)/2}$

Therefore, we have  $P(n)$  for all  $n \geq 0$  by the principle of induction.

# Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that  $f(n) \geq 2^{n/2}$  for all  $n \geq 2$  by induction.

Define  $P(n)$  to be " $f(n) \geq 2^{n/2}$ " We show  $P(n)$  is true for all  $n \geq 2$  by induction on  $n$ .

Base Cases:  $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$$

Inductive Hypothesis: Suppose  $P(2) \wedge P(3) \wedge \dots \wedge P(k)$  for an arbitrary  $k \geq 3$ .

Inductive step:  $f(k+1) = f(k) + f(k-1)$  by the definition of the Fibonacci numbers. Applying IH twice, we have

$$f(k+1) \geq 2^{k/2} + 2^{(k-1)/2}$$

$$\geq 2^{(k+1)/2}$$

Therefore, we have  $P(n)$  for all  $n \geq 0$  by the principle of induction.



# Fibonacci Inequality Two

$$f(0) = 1; \quad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that  $f(n) \geq 2^{n/2}$  for all  $n \geq 2$  by induction.

Define  $P(n)$  to be " $f(n) \geq 2^{n/2}$ " We show  $P(n)$  is true for all  $n \geq 2$  by induction on  $n$ .

Base Cases:  $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

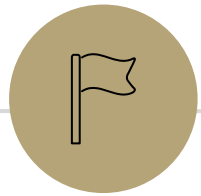
$$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$$

Inductive Hypothesis: Suppose  $P(2) \wedge P(3) \wedge \dots \wedge P(k)$  for an arbitrary  $k \geq 3$ .

Inductive step:  $f(k+1) = f(k) + f(k-1)$  by the definition of the Fibonacci numbers. Applying IH twice, we have

$$\begin{aligned} f(k+1) &\geq 2^{k/2} + 2^{(k-1)/2} \\ &= 2^{(k-1)/2}(\sqrt{2} + 1) \\ &\geq 2^{(k-1)/2} \cdot 2 \\ &\geq 2^{(k+1)/2} \end{aligned}$$

Therefore, we have  $P(n)$  for all  $n \geq 0$  by the principle of induction.



**More Practice**

---

# Even More Induction Practice

$$\text{Let } g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases}$$

$$\text{Let } h(n) = n^n$$

Claim:  $h(n) \geq g(n)$  for all integers  $n \geq 1$

# Even More Induction Practice

Define  $P(n)$  to be " $h(n) \geq g(n)$  for all integers  $n \geq 1$ "

We show  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

Base Case

Inductive Hypothesis:

Inductive Step:

Thus  $P(k + 1)$  holds.

Therefore, we have  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

$$\begin{aligned} \text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n \end{aligned}$$

# Even More Induction Practice

Define  $P(n)$  to be " $h(n) \geq g(n)$  for all integers  $n \geq 1$ "

We show  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

Base Case ( $n = 1$ ):  $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$ .

Inductive Hypothesis: Suppose  $P(k)$  is true for an arbitrary  $k \geq 1$ .

Inductive Step:

$$g(k + 1) = (k + 1) \cdot g(k)$$

$$= (k + 1)^{k+1}.$$

Thus  $P(k + 1)$  holds.

Therefore, we have  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

$$\begin{aligned} \text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n \end{aligned}$$

# Even More Induction Practice

Define  $P(n)$  to be " $h(n) \geq g(n)$  for all integers  $n \geq 1$

We show  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

Base Case ( $n = 1$ ):  $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$ .

Inductive Hypothesis: Suppose  $P(k)$  is true for an arbitrary  $k \geq 1$ .

Inductive Step:

$$\begin{aligned} g(k+1) &= (k+1) \cdot g(k) \\ &\leq (k+1) \cdot h(k) \text{ by IH.} \end{aligned}$$

$$= (k+1)^{k+1}.$$

Thus  $P(k+1)$  holds.

Therefore, we have  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

$$\begin{aligned} \text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n \end{aligned}$$

# Even More Induction Practice

Define  $P(n)$  to be " $h(n) \geq g(n)$  for all integers  $n \geq 1$ "

We show  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

Base Case ( $n = 1$ ):  $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$ .

Inductive Hypothesis: Suppose  $P(k)$  is true for an arbitrary  $k \geq 1$ .

Inductive Step:

$$\begin{aligned}g(k + 1) &= (k + 1) \cdot g(k) \\ &\leq (k + 1) \cdot h(k) && \text{by IH.} \\ &\leq (k + 1) \cdot k^k && \text{by definition of } h(k) \\ &\leq (k + 1) \cdot (k + 1)^k \\ &= (k + 1)^{k+1}.\end{aligned}$$

Thus  $P(k + 1)$  holds.

Therefore, we have  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

$$\begin{aligned}\text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n\end{aligned}$$

# Even More Induction Practice

Define  $P(n)$  to be " $h(n) \geq g(n)$  for all integers  $n \geq 1$ "

We show  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

Base Case ( $n = 1$ ):  $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$ .

Inductive Hypothesis: Suppose  $P(k)$  is true for an arbitrary  $k \geq 1$ .

Inductive Step:

$$\begin{aligned}g(k + 1) &= (k + 1) \cdot g(k) \\ &\leq (k + 1) \cdot h(k) && \text{by IH.} \\ &\leq (k + 1) \cdot k^k && \text{by definition of } h(k) \\ &\leq (k + 1) \cdot (k + 1)^k \\ &= (k + 1)^{k+1}.\end{aligned}$$

Thus  $P(k + 1)$  holds.

Therefore, we have  $P(n)$  for all  $n \geq 1$  by induction on  $n$ .

$$\begin{aligned}\text{Let } g(n) &= \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n - 1) & \text{otherwise} \end{cases} \\ \text{Let } h(n) &= n^n\end{aligned}$$



# Even More Induction Practice: Sums

Let  $P(n)$  be  $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show  $P(n)$  for all  $n \in \mathbb{N}$  by induction on  $n$ .

Base Case ( $n = 0$ ):

Inductive Hypothesis:

Inductive Step:

[Conclusion]

# Even More Induction Practice: Sums

Let  $P(n)$  be  $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show  $P(n)$  for all  $n \in \mathbb{N}$  by induction on  $n$ .

Base Case ( $n = 0$ ):  $\sum_{i=0}^0 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0 + 4)}{2}$

Inductive Hypothesis: Suppose  $P(k)$  is true for an arbitrary  $k \geq 0$ .

Inductive Step:

Target:  $\sum_{i=0}^{k+1} 2 + 3i = \frac{([k+1]+1)(3[k+1]+4)}{2}$

# Even More Induction Practice: Sums

Let  $P(n)$  be  $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show  $P(n)$  for all  $n \in \mathbb{N}$  by induction on  $n$ .

Base Case ( $n = 0$ ):  $\sum_{i=0}^0 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0 + 4)}{2}$

Inductive Hypothesis: Suppose  $P(k)$  is true for an arbitrary  $k \geq 0$ .

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^k 2 + 3i) + (2 + 3(k + 1))$ . By IH, we have:

$$\sum_{i=0}^{k+1} 2 + 3i = \frac{(k+1)(3k+4)}{2} + 2 + 3k + 3 = \text{????}$$

$$= \frac{([k + 1] + 1)(3[k + 1] + 4)}{2}$$

# Even More Induction Practice: Sums

Let  $P(n)$  be  $\sum_{i=0}^n 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show  $P(n)$  for all  $n \in \mathbb{N}$  by induction on  $n$ .

Base Case ( $n = 0$ ):  $\sum_{i=0}^0 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3 \cdot 0 + 4)}{2}$

Inductive Hypothesis: Suppose  $P(k)$  is true for an arbitrary  $k \geq 0$ .

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^k 2 + 3i) + (2 + 3(k + 1))$ . By IH, we have:

$$\begin{aligned} \sum_{i=0}^{k+1} 2 + 3i &= \frac{(k+1)(3k+4)}{2} + 2 + 3k + 3 = \frac{3k^2+7k+4}{2} + \frac{6k+10}{2} = \frac{3k^2+13k+14}{2} = \\ &= \frac{(3k+7)(k+2)}{2} = \frac{([k+1]+1)(3[k+1]+4)}{2} \end{aligned}$$

Therefore,  $P(n)$  holds for all  $n \in \mathbb{N}$  by induction on  $n$ .