Negating Quantifiers, Direct Proof Rule

CSE 311 Spring 2022
Lecture 7

Announcements

Optional reading on domain restriction on the calendar for today's lecture.

Goes through an example more slowly and explains why the rules are what they are.

About Grades

Grades were critical in your lives up until now.

- If you were in high school, they're critical for getting into college.
- If you were at UW or CC applying to CSE, they were key to that application

Regardless of where you're going next, what you **learn** in this course matters FAR more than what your grade is in this course.

If you're planning on industry – interviews matter more than grades.

If you're planning on grad school – letters matter most, those are based on doing work outside of class building off what you learned in class.

About Grades

What that means:

The TAs and I are going to prioritize your learning over debating whether -2 or -1 is “ more fair”

If you’re worried about “have I explained enough” – write more!

It’ll take you longer to write the Ed question than write the extended answer. We don’t take off for too much work.

And the extra writing is going to help you learn more anyway.

Regrades

TAs make mistakes!

When I was a TA, I made errors on 1 or 2% of my grading that needed to be corrected. If we made a mistake, file a regrade request on gradescope.

But those are only for mistakes, not for whether "-1 would be more fair"

If you are confused, please talk to us!

My favorite office hours questions are "can we talk about the best way to do something on the homework we just got back?"

If **after** you do a regrade request on gradescope, you still think a grading was incorrect, send email to Robbie.

Regrade requests will close about 1 week after homework is returned.

Today

Continuing two threads:

Quantifiers

Inference Proofs

At the end (maybe) Inference Proofs with quantifiers

Negating Quantifiers

What happens when we negate an expression with quantifiers?

What does your intuition say?

Original	Negation
Every positive integer is prime	There is a positive integer that is not prime.
$\forall x \ \texttt{Prime}(x)$ Domain of discourse: positive integers	$\exists x(\neg \ \texttt{Prime}(x))$ Domain of discourse: positive integers

Negating Quantifiers

Let's try on an existential quantifier…

Original	Negation
There is a positive integer which is prime and even.	Every positive integer is composite or odd.
$\exists x(\texttt{Prime}(x) \land \texttt{Even}(x))$	$\forall x(\neg\texttt{Prime}(x) \lor \neg\texttt{Even}(x))$
Domain of discourse: positive integers	Domain of discourse: positive integers

To negate an expression with a quantifier
1. Switch the quantifier ($\forall$ becomes $\exists$, $\exists$ becomes $\forall$)
2. Negate the expression inside

Negating Quantifiers

You can think of these negations as applications of DeMorgan's Laws.

Let your domain of discourse be the set containing $d_1, d_2, \ldots, d_n$.

$\exists x(P(x))$ is equivalent to $P(d_1) \vee P(d_2) \vee \cdots \vee P(d_n)$

$\forall x(P(x))$ is equivalent to $P(d_1) \wedge P(d_2) \wedge \cdots \wedge P(d_n)$

Since negating flips ANDs with ORs, it also flips $\exists$ with $\forall$.

Negation

Negate these sentences in English and translate the original and negation to predicate logic.

All cats have nine lives.

$$\forall x\big(Cat(x) \rightarrow NumLives(x,9)\big)$$

$\exists x(Cat(x) \wedge \neg\big(NumLives(x,9)\big))$ "There is a cat without 9 lives."

All dogs love every person.

$$\forall x \forall y\big(Dog(x) \wedge Human(y) \rightarrow Love(x,y)\big)$$

$\exists x \exists y(Dog(x) \wedge Human(y) \wedge \neg Love(x,y))$ "There is a dog who does not love someone." "There is a dog and a person such that the dog doesn't love that person."

There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \wedge Human(y) \wedge Love(x,y)$$

$$\forall x \forall y([Cat(x) \wedge Human(y)] \rightarrow \neg Love(x,y))$$

"For every cat and every human, the cat does not love that human."

"Every cat does not love any human" ("no cat loves any human")

Negation with Domain Restriction

$\exists x \exists y (Cat(x) \wedge Human(y) \wedge Love(x, y))$

$\forall x \forall y ([Cat(x) \wedge Human(y)] \rightarrow \neg Love(x, y))$

There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?

There's a problem in this week's section handout showing similar algebra.

Domain Restriction

Why are the rules what they are?

A universal quantifier is a "Big AND"

For a domain of discourse of $\{e_1, e_2, \dots, e_k\}$

$\forall x(P(x))$ means $P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k)$

Now let's say our domain is $\{e_1, e_2, \dots, e_k, f_1, f_2, \dots, f_j\}$ where f_i are the irrelevant parts of the bigger domain (non-cat-mammals). We want the expression to be

$P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k) \wedge T \wedge T \dots \wedge T$

$\forall x(RightSubDomain(x) \rightarrow P(x))$ does that!

Why are the rules what they are?

An existential quantifier is a “Big OR”

For a domain of discourse of $\{e_1, e_2, \ldots, e_k\}$

$\exists x(P(x))$ means $P(e_1) \vee P(e_2) \vee \cdots \vee P(e_k)$

Now let's say our domain is $\{e_1, e_2, \ldots, e_k, f_1, f_2, \ldots, f_j\}$ where f_i are the irrelevant parts of the bigger domain (non-cat-mammals). We want the expression to be

$P(e_1) \vee P(e_2) \vee \cdots \vee P(e_k) \vee F \vee F \ldots \vee F$

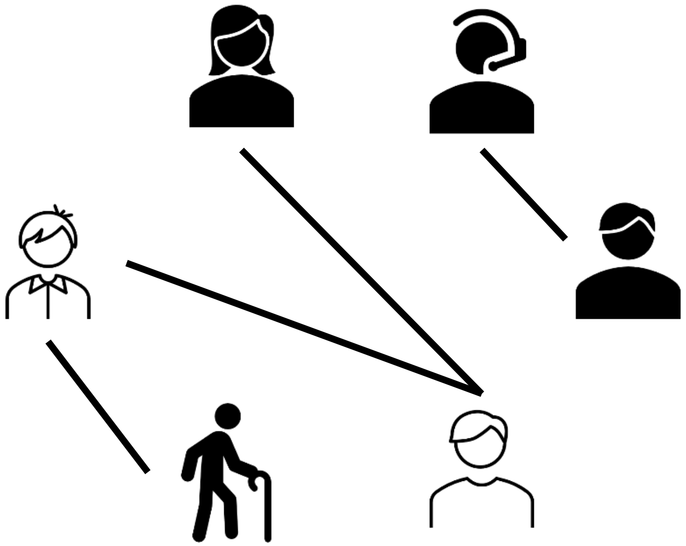
$\exists x(RightSubDomain(x) \wedge P(x))$ does that!

Nested Quantifiers

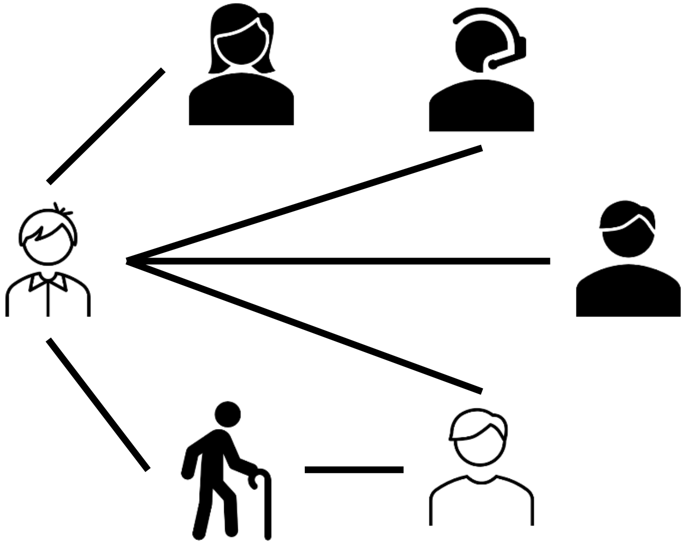
Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x,y)

Everyone is friends with someone.



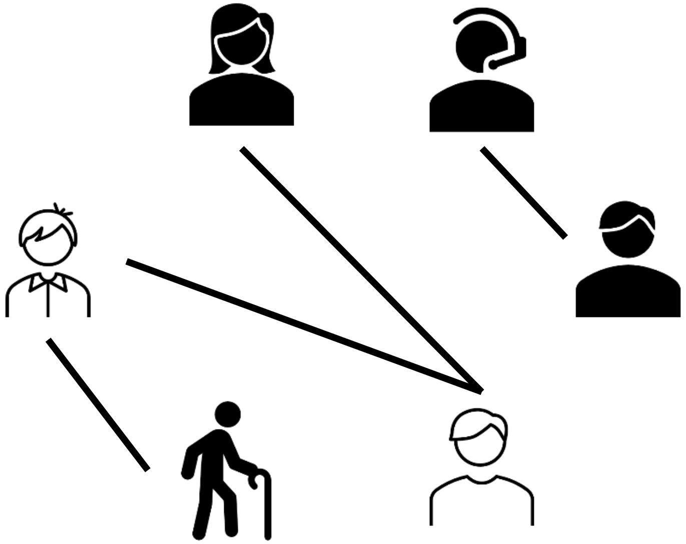
Someone is friends with everyone.



Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

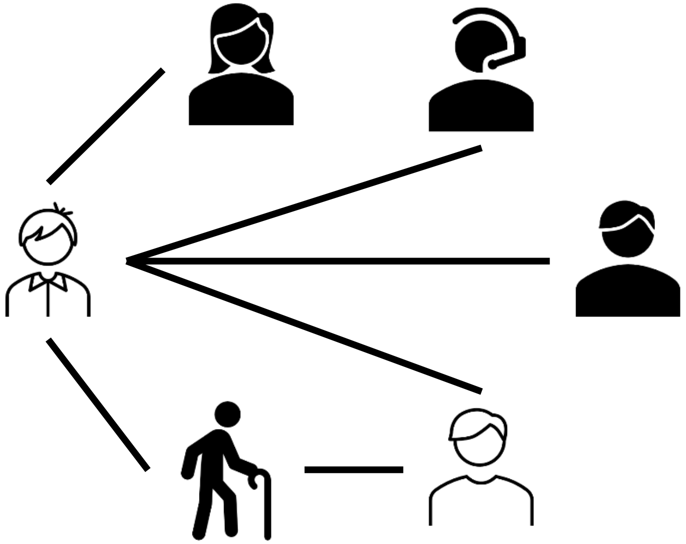
Everyone is friends with someone.



$\forall x (\exists y \texttt{ AreFriends}(x, y))$

$\forall x \exists y \texttt{ AreFriends}(x, y)$

Someone is friends with everyone.



$\exists x (\forall y \texttt{ AreFriends}(x, y))$

$\exists x \forall y \texttt{ AreFriends}(x, y)$

Nested Quantifiers

$\forall x \exists y\, P(x, y)$

"For every x there exists a y such that $P(x, y)$ is true."

y might change depending on the x (people have different friends!).

$\exists x \forall y\, P(x, y)$

"There is an x such that for all $y, P(x, y)$ is true."

There's a special, magical x value so that $P(x, y)$ is true regardless of y.

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

$\forall x \exists y P(x, y)$

y

x

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

A row, where every entry is T

$\forall x \exists y P(x, y)$

In every row there must be a T

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Keep everything in order

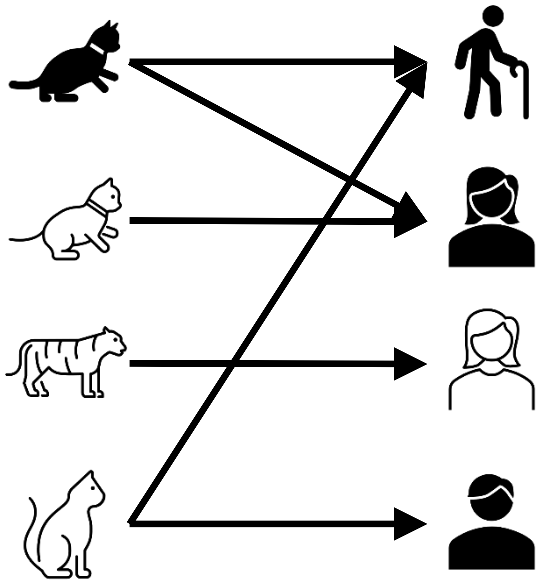
Keep the quantifiers in the same order in English as they are in the logical notation.

“There is someone out there for everyone” is a $\forall x \exists y$ statement in “everyday” English.

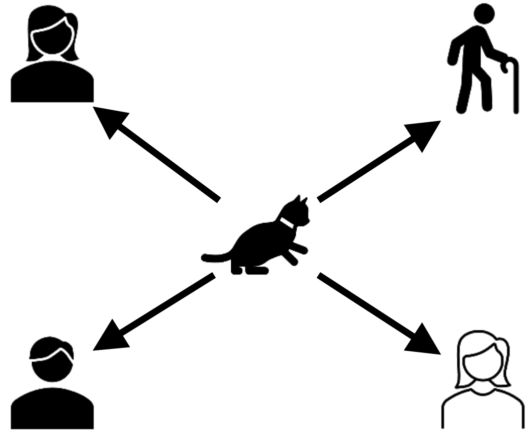
It would **never** be phrased that way in “mathematical English” We’ll only ever write “for every person, there is someone out there for them.”

Try it yourselves

Every cat loves some human.



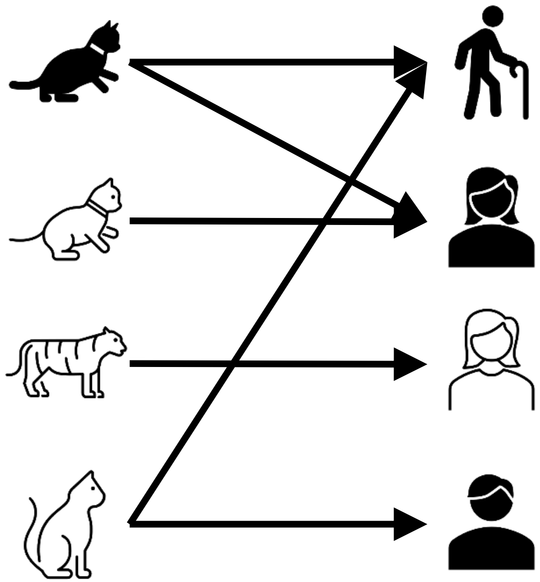
There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates $\texttt{Cat}(x)$, $\texttt{Dog}(x)$, and $\texttt{Loves}(x, y)$ to mean x loves y.

Try it yourselves

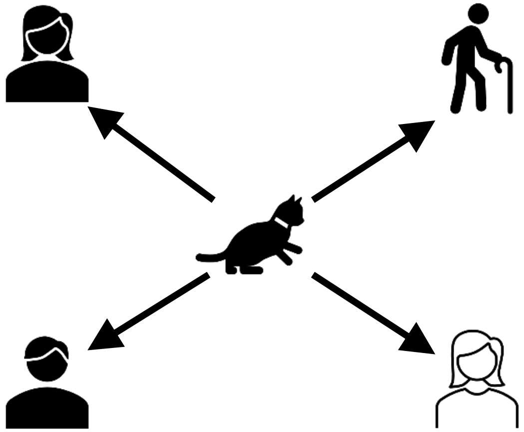
Every cat loves some human.



$\forall x\ (\mathtt{Cat}(x) \to \exists y[\mathtt{Human}(y)\ \land \mathtt{Loves}(x,y)])$

$\forall x \exists y (\mathtt{Cat}(x) \to [\mathtt{Human}(y) \land \mathtt{Loves}(x,y)])$

There is a cat that loves every human.



$\exists x\ (\mathtt{Cat}(x) \land \forall y[\mathtt{Human}(y) \to \mathtt{Loves}(x,y)])$

$\exists x \forall y (\mathtt{Cat}(x) \land [\mathtt{Human}(y) \to \mathtt{Loves}(x,y)])$

Negation

How do we negate nested quantifiers?

The old rule still applies.

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$$\neg(\forall x \exists y \forall z\, [P(x,y) \wedge Q(y,z)])$$

$$\exists x(\neg(\exists y \forall z\, [P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y(\neg(\forall z[P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y \exists z(\neg[P(x,y) \wedge Q(y,z)])$$

$$\exists x \forall y \exists z[\neg P(x,y) \vee \neg Q(y,z)]$$

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y(\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y(\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x(\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$

(This statement is true, y can depend on x

Inference Proofs and the Direct Proof Rule

Try it yourselves

Suppose you know $p \rightarrow q, \neg s \rightarrow \neg q$,and p.
Give an argument to conclude s.

1.	$p \rightarrow q$	Given
2.	$\neg s \rightarrow \neg q$	Given
3.	p	Given
4.	q	Modus Ponens 1,3
5.	$q \rightarrow s$	Contrapositive of 2
6.	s	Modus Ponens 5,4

More Inference Rules

In total, we have two for ∧ and two for ∨, one to create the connector, and one to remove it.

Eliminate ∧

$$\frac{A \land B}{\therefore \quad A, B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \land B}$$

Eliminate ∨

$$\frac{A \lor B, \neg A}{\therefore \quad B}$$

Intro ∨

$$\frac{A}{\therefore A \lor B, B \lor A}$$

None of these rules are surprising, but they are useful.

The Direct Proof Rule

We've been implicitly using another "rule" today, the direct proof rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \rightarrow B}$$

$$\boxed{\text{Direct Proof rule}}\quad \frac{A \Rightarrow B}{A \rightarrow B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…starting right now.

How would you argue…

Let’s say you have a piece of code.

And you think **if** the code gets null input **then** a nullPointerExecption will be thrown.

How would you convince your friend?

You’d probably trace the code, assuming you would get null input.

The code was your **given**

The null input is an assumption

In general

How do you convince someone that $p \rightarrow q$ is true given some surrounding context/some surrounding givens?

You suppose p is true (you assume p)

And then you'll show q must also be true.

Just from p and the Given information.

The Direct Proof Rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \rightarrow B}$$

Direct Proof rule $$\frac{A \Rightarrow B}{A \rightarrow B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…starting today!

Given: $((p \rightarrow q) \wedge (q \rightarrow r))$
Show: $(p \rightarrow r)$

Here's an incorrect proof.

1.	$(p \rightarrow q) \wedge (q \rightarrow r)$	Given
2.	$p \rightarrow q$	Eliminate $\wedge$ (1)
3.	$q \rightarrow r$	Eliminate $\wedge$ (1)
4.	p	Given???
5.	q	Modus Ponens 4,2
6.	r	Modus Ponens 5,3
7.	$p \rightarrow r$	Direct Proof Rule

Given: $((p \to q) \wedge (q \to r))$
Show: $(p \to r)$

Here's an incorrect proof.

1.	$(p \to q) \wedge (q \to r)$	
2.	$p \to q$	
3.	$q \to r$	Eliminate $\wedge$ (1)
4.	p	Given ????
5.	q	Modus Ponens 4,2
6.	r	Modus Ponens 5,3
7.	$p \to r$	Direct Proof Rule

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts...

These facts depend on p.
But p isn't known generally.
It was assumed for the purpose of proving $p \to r$.

Given: $((p \to q) \land (q \to r))$
Show: $(p \to r)$

Here's an incorrect proof.

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts...

1. $(p \to q) \land (q \to r)$
2. $p \to q$
3. $q \to r$ — Eliminate ∧ (1)
4. p — Given ????
5. q — Modus Ponens 4,2
6. r — Modus Ponens 5,3
7. $p \to r$ — Direct Proof Rule

These facts depend on p.
But p isn't known generally.
It was assumed for the purpose of proving $p \to r$.

Given: $((p \to q) \land (q \to r))$
Show: $(p \to r)$

Here's a corrected version of the proof.

1.	$(p \to q) \land (q \to r)$	Given
2.	$p \to q$	Eliminate ∧ 1
3.	$q \to r$	Eliminate ∧ 1
4.1	p	Assumption
4.2	q	Modus Ponens 4.1,2
4.3	r	Modus Ponens 4.2,3
5.	$p \to r$	Direct Proof Rule

When introducing an assumption to prove an implication: Indent, and change numbering.

When reached your conclusion, use the Direct Proof Rule to observe the implication is a fact.

The conclusion is an unconditional fact (doesn't depend on p) so it goes back up a level

Try it!

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

Eliminate ∧	$$\frac{A \wedge B}{\therefore \quad A, B}$$
Eliminate ∨	$$\frac{A \vee B, \neg A}{\therefore \quad B}$$
Intro ∧	$$\frac{A; B}{\therefore \quad A \wedge B}$$
Intro ∨	$$\frac{A}{\therefore A \vee B, B \vee A}$$
Direct Proof rule	$$\frac{A \Rightarrow B}{A \rightarrow B}$$
Modus Ponens	$$\frac{P \rightarrow Q; P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

Try it!

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

1.	$p \vee q$	Given
2.	$(r \wedge s) \rightarrow \neg q$	Given
3.	r	Given
4.1	s	Assumption
4.2	$r \wedge s$	Intro $\wedge$ (3,4.1)
4.3	$\neg q$	Modus Ponens (2, 4.2)
4.4	$q \vee p$	Commutativity (1)
4.5	p	Eliminate $\vee$ (4.4, 4.3)
5.	$s \rightarrow p$	Direct Proof Rule

Inference Rules

Eliminate ∧
$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨
$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧
$$\frac{A; B}{\therefore \quad A \wedge B}$$

Intro ∨
$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule
$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens
$$\frac{P \rightarrow Q; P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

Inference Proofs in Predicate Logic

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀ $\dfrac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$

Intro ∃ $\dfrac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$

Intro ∀ $\dfrac{P(a);\, a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$

Eliminate ∃ $\dfrac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$

Let's see a good example, then come back to those "arbitrary" and "fresh" conditions.

Proof Using Quantifiers

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \rightarrow Q(y)]$. Conclude $\exists x Q(x)$.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\quad \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore\qquad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore\ \ P(c) \text{ for a fresh } c}$$

Proof Using Quantifiers

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \to Q(y)]$. Conclude $\exists x Q(x)$.

Intro $\exists$

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Eliminate $\exists$

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

Eliminate $\forall$

$$\frac{\forall x\, P(x)}{\therefore\, P(a) \text{ for any } a}$$

Intro $\forall$

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore \quad \forall x\, P(x)}$$

Proof Using Quantifiers

Suppose we know $\exists x P(x)$ and $\forall y[\, P(y) \to Q(y)]$. Conclude $\exists x Q(x)$.

1.	$\exists x P(x)$	Given
2.	$P(a)$	Eliminate $\exists$ 1
3.	$\forall y[P(y) \to Q(y)]$	Given
4.	$P(a) \to Q(a)$	Eliminate $\forall$ 3
5.	$Q(a)$	Modus Ponens 2,4
6.	$\exists x Q(x)$	Intro $\exists$ 5

Intro $\exists$

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Eliminate $\exists$

$$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$$

Eliminate $\forall$

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro $\forall$

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore\ \forall x\, P(x)}$$

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore \qquad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore \quad P(c) \text{ for a fresh } c}$$

"arbitrary" means a is "just" a variable in our domain.
It doesn't depend on any other variables and wasn't introduced with other information.

Proofs with Quantifiers

We've done symbolic proofs with propositional logic.

To include predicate logic, we'll need some rules about how to use quantifiers.

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Intro ∀

$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore\ \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$$

"fresh" means c is a new symbol (there isn't another c somewhere else in our proof).

Fresh and Arbitrary

Suppose we know $\exists x P(x)$. Can we conclude $\forall x P(x)$?

1. $\exists x\, P(x)$ Given
2. $P(a)$ Eliminate $\exists$ (1)
3. $\forall x\, P(x)$ Intro $\forall$ (2)

This proof is **definitely** wrong.
(take $P(x)$ to be "is a prime number")

a wasn't **arbitrary**. We knew something about it – it's the x that exists to make $P(x)$ true.

Intro $\exists$
$$\frac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$$

Eliminate $\exists$
$$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$$

Eliminate $\forall$
$$\frac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$$

Intro $\forall$
$$\frac{P(a);\, a \text{ is arbitrary}}{\therefore\ \forall x\, P(x)}$$

Fresh and Arbitrary

Intro ∀

$$\frac{P(a);\ a \text{ is arbitrary}}{\therefore\quad \forall x\, P(x)}$$

Eliminate ∃

$$\frac{\exists x P(x)}{\therefore\quad P(c) \text{ for a fresh } c}$$

You can trust a variable to be **arbitrary** if you introduce it as such.

If you eliminated a ∀ to create a variable, that variable is arbitrary. Otherwise it's not arbitrary – it depends on something.

You can trust a variable to be **fresh** if the variable doesn't appear anywhere else (i.e. just use a new letter)

Fresh and Arbitrary

Eliminate ∀

$$\frac{\forall x\, P(x)}{\therefore P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

There are no similar concerns with these two rules.

Want to reuse a variable when you eliminate ∀? Go ahead.

Have a c that depends on many other variables, and want to intro ∃?

Also not a problem.

Arbitrary

In section, you said: $[\exists y \forall x\ P(x, y)] \rightarrow [\forall x \exists y\ P(x, y)]$. Let's prove it!!

Arbitrary

In section, you said: $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\, P(x,y)$	Assumption
1.2 $\forall x\, P(x,c)$	Elim $\exists$ (1.1)
1.3 Let a be arbitrary.	--
1.4 $P(a,c)$	Elim $\forall$ (1.2)
1.5 $\exists y\, P(a,y)$	Intro $\exists$ (1.4)
1.6 $\forall x \exists y\, P(x,y)$	Intro $\forall$ (1.5)

2. $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$ Direct Proof Rule

Arbitrary

In section, you said: $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$. Let's prove it!!

1.1 $\exists y \forall x\, P(x,y)$	Assumption
1.2 $\forall x\, P(x,c)$	Elim $\exists$ (1.1)
1.4 $P(a,c)$	Elim $\forall$ (1.2)
1.5 $\exists y\, P(a,y)$	Intro $\exists$ (1.4)
1.6 $\forall x \exists y\, P(x,y)$	Intro $\forall$ (1.5)

2. $[\exists y \forall x\, P(x,y)] \rightarrow [\forall x \exists y\, P(x,y)]$ Direct Proof Rule

It is not required to have "variable is arbitrary" as a step before using it. But many people (including Robbie) find it helpful.

Find The Bug

Let your domain of discourse be integers.
We claim that given $\forall x \exists y$ Greater(y, x), we can conclude $\exists y \forall x$ Greater(y, x)
Where Greater(y, x) means $y > x$

1.	$\forall x \exists y$ Greater(y, x)	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y$ Greater(y, a)	Elim $\forall$ (1)
4.	$b \geq a$	Elim $\exists$ (2)
5.	$\forall x$ Greater(b, x)	Intro $\forall$ (4)
6.	$\exists y \forall x$ Greater(y, x)	Intro $\exists$ (5)

Find The Bug

1.	$\forall x \exists y\ \texttt{Greater}(y, x)$	Given
2.	Let a be an arbitrary integer	--
3.	$\exists y\ \texttt{Greater}(y, a)$	Elim ∀ (1)
4.	$b \geq a$	Elim ∃ (2)
5.	$\forall x\ \texttt{Greater}(b, x)$	Intro ∀ (4)
6.	$\exists y \forall x\ \texttt{Greater}(y, x)$	Intro ∃ (5)

b is not arbitrary. The variable b depends on a. Even though a is arbitrary, b is not!

Bug Found

There's one other "hidden" requirement to introduce $\forall$.

"No other variable in the statement can depend on the variable to be generalized"

Think of it like this -- b was probably $a + 1$ in that example.

You wouldn't have generalized from $\texttt{Greater}(a + 1, a)$

To $\forall x\ \texttt{Greater}(a + 1, x)$. There's still an a, you'd have replaced all the a's.

x depends on y if y is in a statement when x is introduced.

This issue is much clearer in English proofs, which we'll start next time.

More Practice

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Show if we know: $p, q, [(p \wedge q) \rightarrow (r \wedge s)], r \rightarrow t$ we can conclude t.

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1.	p	Given
2.	q	Given
3.	$[(p \wedge q) \rightarrow (r \wedge s)]$	Given
4.	$r \rightarrow t$	Given
5.	$p \wedge q$	Intro ∧ (1,2)
6.	$r \wedge s$	Modus Ponens (3,5)
7.	r	Eliminate ∧ (6)
8.	t	Modus Ponens (4,7)