Recursive Definitions of Sets

Basis: $6 \in S, 15 \in S$

Recursive: If $x, y \in S$ then $x + y \in S$

Basis: $\begin{bmatrix}1\\1\\0\end{bmatrix} \in S$ $\begin{bmatrix}1\\0\\1\end{bmatrix} \in S$

Recursive: if $x \in S$, then $\alpha x \in S$ for all $\alpha \in \mathbb{R}$.

If $x, y \in S$ then $x + y \in S$.

Write a recursive definition of $\{x: x = 3^i$ for some $i \in \mathbb{N}\}$.

Functions on Strings

Since strings are defined recursively, most functions on strings are as well.

Length:

len(ε)=0;

len(wa)=len(w)+1 for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$;
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation

$x \cdot \varepsilon = x$ for all $x \in \Sigma^*$;
$x \cdot (wa) = (x \cdot w)a$ for $w \in \Sigma^*, a \in \Sigma$

Number of c's in a string

$\#_c(\varepsilon) = 0$
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$;
$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma \setminus \{c\}$.

Structural Induction

Let $P(x)$ be "x is divisible by 3."

We show $P(x)$ holds for all $x \in S$ by structural induction.

Base Cases:
Inductive Hypothesis:

Inductive Step:

We conclude $P(x) \forall x \in \mathrm{S}$ by the principle of induction.

Basis: $6 \in \mathrm{S}, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

Claim len(x·y)=len(x) + len(y) for all $x, y \in \Sigma^*$.

Define Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.

Base Case:

Inductive Hypothesis:

Inductive Step:

Σ^*:Basis: $\varepsilon \in \Sigma^*$.
Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$