Irregular Languages

CSE 311 Spring 2022
Lecture 27

What can't NFAs/DFAs/Regexes do?

There are languages that can't be recognized by a DFA.

$\{0^k 1^k | k \geq 0\}$ is the classic example.

Let's prove it.

$\{0^k 1^k | k \geq 0\}$ is not regular

Not a proof:

$\{0^k 1^k | k \geq 0\}$ is regular if and only if there's a DFA the recognizes it.

To know if the number of 0's is equal to the number of 1's that DFA is going to have to count that number k.

But that number could be arbitrarily big.

And we have some finite number of states less than that.

So it's not regular!

"the DFA must work like this" isn't a rigorous argument.

“The DFA must do this”

Consider “The set of all binary strings, where the number of “01” substrings is equal to the number of “10” substrings.”

The DFA must count the number of “01” and “10” substrings, right?

NO!

Between every 01 substring there’s a 10 substring.

00011110000111101011

You don’t have to count the total, just the difference between them. And that difference never exceeds 1.

In general…

Try to avoid saying "in order to solve problem X, a machine must do Y"

It's almost impossible to rigorously justify.

And it's easy to trick yourself – you'll have to argue no one is more clever than you are.

The right way to argue around this claim is to argue by contradiction. Show that every conceivable DFA does the wrong thing (rather than indirectly that they aren't doing what you think they should).

A Proof Outline

Claim: $\{0^k 1^k : k \geq 0\}$ is an irregular language.

Proof:

Suppose, for the sake of contradiction, that $\{0^k 1^k : k \geq 0\}$ is regular.

Then there is a DFA M such that M accepts exactly $\{0^k 1^k : k \geq 0\}$.

We want to show M ***doesn't*** *actually accept* $\{0^k 1^k : k \geq 0\}$

How do we do that?

Forcing a Mistake

We don't know anything about the DFA.

Well except that it **is** a DFA.

So it is deterministic. I.e. if you're in a particular state and you read a certain character there's only one place to go.

It's also finite. I.e. there are not an infinite number of states.

Forcing a Mistake

What if…

We could find a set of strings that all **must** be in different states.

Too many of them. Enough that two must be in the same state.

How would we know two strings have to be in different states?

How many strings is enough? infinity

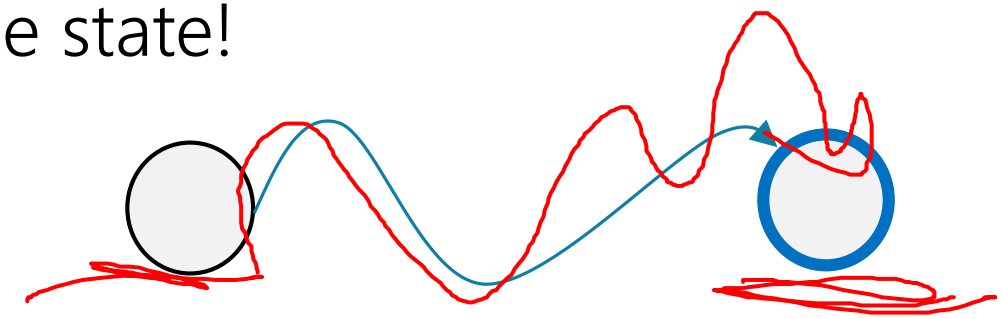
Forcing a Mistake

How do we know x, y must be in different states?

Well if one would be accepted and the other rejected, that would be a clear sign.

Or if there's some string z where xz is accepted but yz is rejected (or vice versa).

The machine is deterministic! If x and y take you to the same state, then xz and yz are also in the same state!



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Then there is a DFA M such that M accepts exactly $\{0^k 1^k : k \geq 0\}$.

Let S = [TODO]. *S is an infinite set of strings.*

Because the DFA is finite, there are two (different) strings x, y in S such that x and y go to the same state. *We don't get to choose x, y*

Consider the string z =[TODO] *We do get to choose z* depending on x, y

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Since x, y led to the same state and M is deterministic, xz and yz will also lead to the same state q in M. Observe that $xz \in \{0^k 1^k : k \geq 0\}$ but $yz \notin \{0^k 1^k : k \geq 0\}$. Since q is can be only one of an accept or reject state, M does not actually recognize $\{0^k 1^k : k \geq 0\}$. That's a contradiction!

Therefore, $\{0^k 1^k : k \geq 0\}$ is an irregular language.

Filling in the blanks

Need S

Such that

1. S is infinite

2. For every pair of strings x, y in S, there is a suffix z such that one of xz, yz is in the language and the other is not in the language.

Let $S = \{0^k : k \geq 0\}$

$x = 0^a$, $y = 0^b$ with $a \neq b$. Take $z = 1^a$.

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Proof:

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Then there is a DFA M such that M accepts exactly $\{0^k 1^k : k \geq 0\}$.

Let $S = \{0^k : k \geq 0\}$.

Because the DFA is finite and S is infinite, there are two (different) strings x, y in S such that x and y go to the same state when read by M. Since both are in S, $x = 0^a$ for some integer a, and $y = 0^b$ for some integer b, with $a \neq b$.

Consider the string $z = 1^a$. $xz = 0^a 1^a \in \{0^k 1^k : k \geq 0\}$ but $yz = 0^b 1^a \notin \{0^k 1^k : k \geq 0\}$.

Since x, y both end up in the same state, and we appended the same z, both xz and yz end up in the same state of M.

Since $xz \in \{0^k 1^k : k \geq 0\}$ and $yz \notin \{0^k 1^k : k \geq 0\}$, M does not recognize $\{0^k 1^k : k \geq 0\}$. But that's a contradiction!

So $\{0^k 1^k : k \geq 0\}$ must be an irregular language.

Full outline

1. Suppose for the sake of contradiction that L is regular. Then there is some DFA M that recognizes L.

2. Let S be [fill in with an infinite set of prefixes].

3. Because the DFA is finite and S is infinite, there are two (different) strings x, y in S such that x and y go to the same state when read by M *[you don't get to control x, y other than having them not equal and in S]*

4. Consider the string z [argue exactly one of $\mathbf{xz}, \mathbf{yz}$ will be in $\mathbf{L}$]

5. Since x, y both end up in the same state, and we appended the same z, both xz and yz end up in the same state of M. Since $xz \in L$and $yz \notin L$, M does not recognize L. But that's a contradiction!

6. So L must be an irregular language.

Practical Tips

When you're choosing the set S, think about what the DFA would "have to count"

That is fundamentally why a language is irregular. The set S is the way we prove it! Whatever we "need to remember" it's different for every element of S.

If your strings have an "obvious middle" (like between the 0's and 1's) that's a good place to start.

Let’s Try another

The set of strings with balanced parentheses is not regular.

What do you want S to be? What would you have to count?

The number of unclosed parentheses.

Let $S =$…

Let's Try another

The set of strings with balanced parentheses is not regular.

What do you want S to be? What would you have to count?

The number of unclosed parentheses.

Want S to be a set with infinitely many strings with different numbers of unclosed parentheses.

Let $S = (^*$

Outline for $(^*$

1. Suppose for the sake of contradiction that L is regular. Then there is some DFA M that recognizes L.

2. Let S be $(^*$

3. Because the DFA is finite and S is infinite, there are two (different) strings x, y in S such that x and y go to the same state when read by M Observe that $x = (^a$ for some integer a, $y = (^b$ for some integer b with $a \neq b$.

4. Consider the string z [argue exactly one of xz, yz will be in L]

5. Since x, y both end up in the same state, and we appended the same z, both xz and yz end up in the same state of M. Since $xz \in L$and $yz \notin L$, M does not recognize L. But that's a contradiction!

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1. Suppose for the sake of contradiction that L is regular. Then there is some DFA M that recognizes L.

2. Let S be $(^*$

3. Because the DFA is finite and S is infinite, there are two (different) strings x, y in S such that x and y go to the same state when read by M Observe that $x = (^a$ for some integer a, $y = (^b$ for some integer b with $a \neq b$.

4. Consider the string $z =)^a$ xz is a balanced set of parentheses (since there are the same number of each and all the open-parentheses come before the close parentheses). But yz is not balanced because $a \neq b$.

5. Since x, y both end up in the same state, and we appended the same z, both xz and yz end up in the same state of M. Since $xz \in L$and $yz \notin L$, M does not recognize L. But that's a contradiction!

6. So L must be an irregular language.

One more, just the key steps

What about $\{a^k b^k c^k : k \geq 0\}$?

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What about $\{a^k b^k c^k : k \geq 0\}$?

$S = \{a^k : k \geq 0\}$ or $S = \{a^k b^k : k \geq 0\}$ are equally good choices.

Your suffix is $b^j c^j$ for the first and c^j for the second.

$S = \{a^k b : k \geq 0\}$ also works. $z = b^{j-1} c^j$ is your suffix. The proof is a little more tedious, but you can make it through.

One Last Fun Fact

On HW8, the extra credit is to use the “pumping lemma” to prove a language is irregular.

The method from lecture “always works” – i.e. for every non-regular language, you can write a proof like this.

For the pumping lemma that isn’t true – there are some non-regular languages that satisfy the pumping lemma, which is why we don’t focus on it.

The final common way to show languages are regular or not regular is “closure properties” – operations that (when applied to regular languages) always give you another regular language. Or vice versa.