Section 06: Solutions

1. A Horse of a Different Color

Did you know that all dogs are named Dubs? It's true. Maybe. Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n\ P(n)$ by induction on n.

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k.

Inductive Step: Consider an arbitrary group of $k+1$ dogs. Arbitrarily select a dog, D, and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D'. We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k+1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k+1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

Solution:

The bug is in the final sentence of the inductive step. We claimed that the groups overlapped, i.e. that some dog was in both of them. That's true for large k, but not when $k+1=2$. When $k=2$, D is in a group by itself, and D' was in a group by itself. The inductive hypothesis holds (D has the only name in its subgroup, and D' has the only name in its subgroup) but returning to the full group $\{D, D'\}$ we cannot conclude that they share a name.

From there everything unravels. $P(1) \not\to P(2)$, so we cannot use the principle of induction. It turns out this is the **only** bug in the proof. The argument in the inductive step is correct as long as $k+1>2$. But that implication is always vacuous, since $P(2)$ is false.

2. Induction with Divides

Prove that $9 \mid (n^3+(n+1)^3+(n+2)^3)$ for all $n>1$ by induction. **Solution:**

Let $P(n)$ be "$9 \mid n^3+(n+1)^3+(n+2)^3$". We will prove $P(n)$ for all integers $n>1$ by induction.

Base Case $(n=2)$**:** $2^3+(2+1)^3+(2+2)^3 = 8+27+64 = 99 = 9\cdot 11$, so $9 \mid 2^3+(2+1)^3+(2+2)^3$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid j^3+(j+1)^3+(j+2)^3$ for an arbitrary integer $j>1$. Note that this is equivalent to assuming that $j^3+(j+1)^3+(j+2)^3 = 9k$ for some integer k by the definition of divides.

Induction Step: $\boxed{\text{Goal: Show } 9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3}$

$$
\begin{aligned}
(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 \text{ for some integer } k && \text{[Induction Hypothesis]} \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}
$$

Since j is an integer, $j^2 + 3j + 3 + k$ is also an integer. Therefore, by the definition of divides, $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for an arbitrary integer $j > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

3. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$
\begin{aligned}
&f(0) = 0 \\
&f(1) = 1 \\
&f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2
\end{aligned}
$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

Solution:

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n.

Base Cases $(n = 0, n = 1)$**:** $f(0) = 0$ and $f(1) = 1$ by definition.

Inductive Hypothesis: Assume that $P(0) \wedge P(1) \wedge \ldots P(k)$ hold for some arbitrary $k \geq 1$.

Inductive Step: We show $P(k+1)$:

$$
\begin{aligned}
f(k+1) &= 2f(k) - f(k-1) && \text{[Definition of } f\text{]} \\
&= 2(k) - (k-1) && \text{[Induction Hypothesis]} \\
&= k + 1 && \text{[Algebra]}
\end{aligned}
$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by principle of strong induction.

4. Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- `soy`(x) is true iff x contains soy milk.
- `whole`(x) is true iff x contains whole milk.
- `sugar`(x) is true iff x contains sugar
- `decaf`(x) is true iff x is not caffeinated.
- `vegan`(x) is true iff x is vegan.
- `RobbieLikes`(x) is true iff Robbie likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

(a) Coffee drinks with whole milk are not vegan. **Solution:**

$\forall x(\texttt{whole}(x) \to \neg\,\texttt{vegan}(x))$.

(b) Robbie only likes one coffee drink, and that drink is not vegan. **Solution:**

$\exists x \forall y(\texttt{RobbieLikes}(x) \wedge \neg\,\texttt{Vegan}(x) \wedge [\texttt{RobbieLikes}(y) \to x = y])$

OR $\exists x(\texttt{RobbieLikes}(x) \wedge \neg\,\texttt{Vegan}(x) \wedge \forall y[\texttt{RobbieLikes}(y) \to x = y])$

(c) There is a drink that has both sugar and soy milk. **Solution:**

$\exists x(\texttt{sugar}(x) \wedge \texttt{soy}(x))$

Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\texttt{decaf}(x) \wedge \texttt{RobbieLikes}(x)] \to \texttt{sugar}(x))$$

Solution:

Every decaf drink that Robbie likes has sugar.

Statements like "For every decaf drink, if Robbie likes it then it has sugar" are equivalent, but only partially take advantage of domain restriction.

5. Midterm Review: Set Theory

Suppose that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Solution:

Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be an arbitrary element. Then by definition of powerset, $X \subseteq A$. Let $y \in X$ be arbitrary. Then since $X \subseteq A$, by definition of subset, $y \in A$. Since $A \subseteq B$, by definition of subset again, $y \in B$. Since y was arbitrary in X, by definition of subset once more, $X \subseteq B$. Then by definition of powerset, $X \in \mathcal{P}(B)$. Since X was arbitrary in $\mathcal{P}(A)$, we have shown $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

6. Midterm Review: Number Theory

Let p be a prime number at least 3, and let x be an integer such that $x^2 \% p = 1$.

(a) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$. (this proof will be short!) (Try to do this without using the theorem "Raising Congruences To A Power")
Solution:

Let y be an arbitrary integer and suppose $y \equiv 1 \pmod{p}$. We can multiply congruences, so multiplying this congruence by itself we get $y^2 \equiv 1^2 \pmod{p}$. Since y is arbitrary, the claim holds.

(b) Repeat part (a), but don't use any theorems from the Number Theory Reference Sheet. That is, show the claim directly from the definitions.

Solution:

Suppose $x \equiv 1 \pmod{p}$. By the definition of Congruences, $p \mid (x-1)$. Therefore, by the definition of divides, there exists an integer k such that

$$pk = (x-1)$$

By multiplying both sides of pk = (x - 1) by (x + 1) and re-arranging the equation, we have

$$pk(x+1) = (x-1)(x+1)$$
$$p(k(x+1)) = (x-1)(x+1)$$

Since $(x-1)(x+1) = x^2 - 1$, by replacing $(x-1)(x+1)$ with $x^2 - 1$, we have

$$p(k(x+1)) = x^2 - 1$$

Note that since k and x are integers, (k (x + 1)) is also an integer. Therefore, by the definition of divides $p \mid x^2 - 1$.
Hence, by the definition of Congruences, $x^2 \equiv 1 \pmod{p}$.

(c) From part (a), we can see that $x\%p$ can equal 1. Show that for any integer x, if $x^2 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. That is, show that the only value $x\%p$ can take other than 1 is $p-1$.
Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod{p}$ and use the fact that $x^2 - 1 = (x-1)(x+1)$
Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

Solution:

Suppose $x^2 \equiv 1 \pmod{p}$. By the definition of Congruences,

$$p \mid x^2 - 1$$

Since $(x-1)(x+1) = x^2 - 1$, by replacing $x^2 - 1$ with $(x-1)(x+1)$, we have

$$p \mid (x-1)(x+1)$$

Note that for an integer p if p is a prime number and $p \mid (ab)$, then $p \mid a$ or $p \mid b$. In this case, since p is a prime number, by applying the rule, we have $p \mid (x-1)$ or $p \mid (x+1)$.
Therefore, by the definition of Congruences, we have $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

7. Midterm Review: Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Solution:

Let $\mathsf{P}(n)$ be the statement "$S_n = \frac{1}{6}n(n+1)(2n+1)$" defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case: When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $\mathsf{P}(0)$ is true.

Inductive Hypothesis: Suppose that $\mathsf{P}(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step: Examining S_{k+1}, we see that

$$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the inductive hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned}
S_{k+1} &= S_k + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right) \\
&= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$

Thus, we can conclude that $\mathsf{P}(k+1)$ is true.

Conclusion: $P(n)$ holds for all integers $n \geq 0$ by the principle of induction.

8. Midterm Review: Strong Induction

Robbie is planning to buy snacks for the members of his competitive roller-skating troupe. However, his local grocery store sells snacks in packs of 5 and packs of 7.

Prove that Robbie can buy exactly n snacks for all integers $n \geq 24$

Solution:

Let $\mathsf{P}(n)$ be the statement "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" defined for all $n \geq 24$. We prove that $\mathsf{P}(n)$ is true for all $n \geq 24$ by the principle of strong induction.

Base Case:

n = 24: 24 snacks can be bought with 2 packs of 7 and 2 packs of 5 snacks.
n = 25: 25 snacks can be bought with 5 packs of 5 snacks.
n = 26: 26 snacks can be bought with 3 packs of 7 and 1 pack of 5 snacks.
n = 27: 27 snacks can be bought with 1 pack of 7 and 4 packs of 5 snacks.
n = 28: 28 snacks can be bought with 4 packs of 7 snacks.

Inductive Hypothesis: Suppose that $\mathsf{P}(24) \wedge \mathsf{P}(25) \wedge \cdots \wedge \mathsf{P}(k)$ is true for some arbitrary $k \geq 28$.

Inductive Step: We want to show that Robbie can buy exactly $k+1$ snacks. By the inductive hypothesis, we know that Robbie can buy exactly $k-4$ snacks, so he can buy another pack of 5 to get exactly $k+1$ snacks.

Conclusion: Therefore, $\mathsf{P}(n)$ holds for all integers $n \geq 24$ by the principle of strong induction.