Quantifier Negation and Direct Proof

CSE 311: Fall 23
Lecture 7

Slides adapted from Anjali Agarwal

Announcements

HW1 solutions handed out in lecture today.

If you missed them, they are in an envelope outside my office door

CSE2 311---in Gates take the elevator to the 3rd floor, turn left.

Don't need to wait for office hours, upper floors are unlocked 8-5 weekdays

HW2

Gradescope has separate boxes for Extra credit, feedback

Late Days are counted from later of main box, extra credit box submitted. (But it's only one late day to resubmit both).

We don't count late days for the feedback box (please just don't forget to submit).

Quantifiers

Which of these translates "For every cat: if a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\texttt{Cat}(x) \land \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$

For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\texttt{Cat}(x) \land (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[what if x is a dog? Dogs are in the domain, but…uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

Quantifiers

Existential quantifiers need a different rule:

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

There is a dog who is not happy.

Domain of discourse: dogs

$\exists x(\neg\, \texttt{Happy}(x))$

Quantifiers

Which of these translates "There is a dog who is not happy."
when our domain of discourse is "mammals"?

$$\exists x[\texttt{Dog}(x) \rightarrow \neg\texttt{Happy}(x))]$$

There is a mammal, such that if x is a dog then it is not happy.
[this can't be right – plug in a cat for x and the implication is true]

$$\exists x[(\texttt{Dog}(x) \wedge \neg\texttt{Happy}(x)]$$

There is a mammal that is both a dog and not happy.
[this one is correct!]

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

Why are the rules what they are?

A universal quantifier is a "Big AND"

For a domain of discourse of $\{e_1, e_2, \dots, e_k\}$

$\forall x(P(x))$ means $P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k)$

Now let's say our domain is $\{e_1, e_2, \dots, e_k, f_1, f_2, \dots, f_j\}$ where f_i are the irrelevant parts of the bigger domain (non-cat-mammals). We want the expression to be

$P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k) \wedge T \wedge T \dots \wedge T$

$\forall x(RightSubDomain(x) \rightarrow P(x))$ does that!

Why are the rules what they are?

An existential quantifier is a "Big OR"

For a domain of discourse of $\{e_1, e_2, \dots, e_k\}$

$\exists x(P(x))$ means $P(e_1) \vee P(e_2) \vee \cdots \vee P(e_k)$

Now let's say our domain is $\{e_1, e_2, \dots, e_k, f_1, f_2, \dots, f_j\}$ where f_i are the irrelevant parts of the bigger domain (non-dog-mammals). We want the expression to be

$P(e_1) \vee P(e_2) \vee \cdots \vee P(e_k) \vee F \vee F \dots \vee F$

$\exists x(RightSubDomain(x) \wedge P(x))$ does that!

Negating Quantifiers

What happens when we negate an expression with quantifiers?

What does your intuition say?

Original	Negation
Every positive integer is prime	There is a positive integer that is not prime.
$\forall x\ \texttt{Prime}(x)$ Domain of discourse: positive integers	$\exists x(\neg\ \texttt{Prime}(x))$ Domain of discourse: positive integers

Negating Quantifiers

Let's try on an existential quantifier…

Original	Negation
There is a positive integer which is prime and even.	Every positive integer is composite or odd.
$\exists x(\texttt{Prime}(x) \wedge \texttt{Even}(x))$ Domain of discourse: positive integers	$\forall x(\neg\texttt{Prime}(x) \vee \neg\texttt{Even}(x))$ Domain of discourse: positive integers

To negate an expression with a quantifier

1. Switch the quantifier ($\forall$ becomes $\exists$, $\exists$ becomes $\forall$)
2. Negate the expression inside

Negation

Translate these sentences to predicate logic, then negate them.

All cats have nine lives.

$$\forall x\big(Cat(x) \rightarrow NumLives(x,9)\big)$$

$\exists x(Cat(x) \wedge \neg(NumLives(x,9)))$ "There is a cat without 9 lives.

All dogs love every person.

$$\forall x \forall y\big(Dog(x) \wedge Human(y) \rightarrow Love(x,y)\big)$$

$\exists x \exists y(Dog(x) \wedge Human(y) \wedge \neg Love(x,y))$ "There is a dog who does not love someone." "There is a dog and a person such that the dog doesn't love that person."

There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \wedge Human(y) \wedge Love(x,y)$$

$$\forall x \forall y(Cat(x) \wedge Human(y) \rightarrow \neg Love(x,y))$$

"For every cat and every human, the cat does not love that human."
"Every cat does not love any human" ("no cat loves any human")

Negation with Domain Restriction

$\exists x \exists y (Cat(x) \wedge Human(y) \wedge Love(x, y))$

$\forall x \forall y ([Cat(x) \wedge Human(y)] \rightarrow \neg Love(x, y))$

There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?

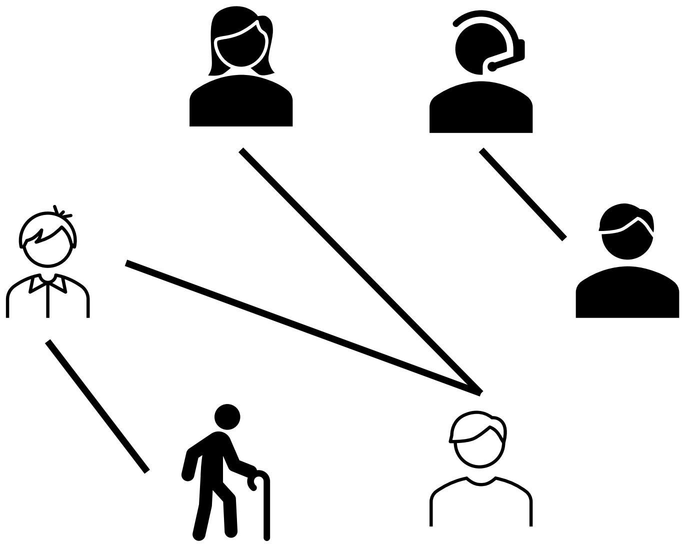
There's a problem in this week's section handout showing similar algebra.

Nested Quantifiers

Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

Everyone is friends with someone.



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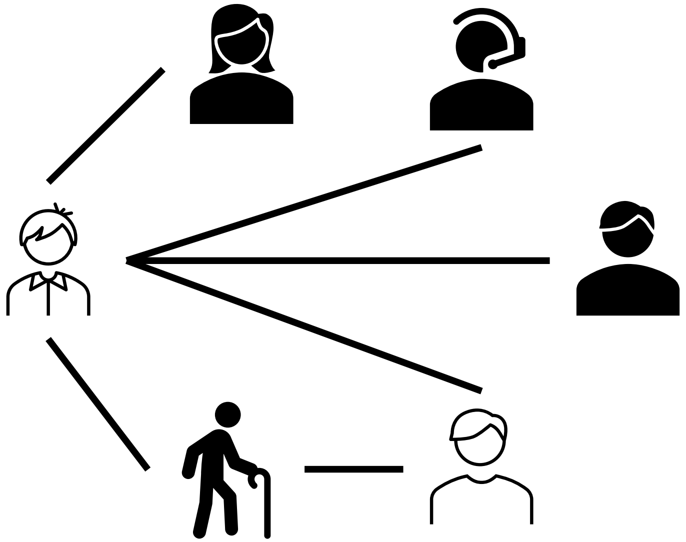

Nested Quantifiers

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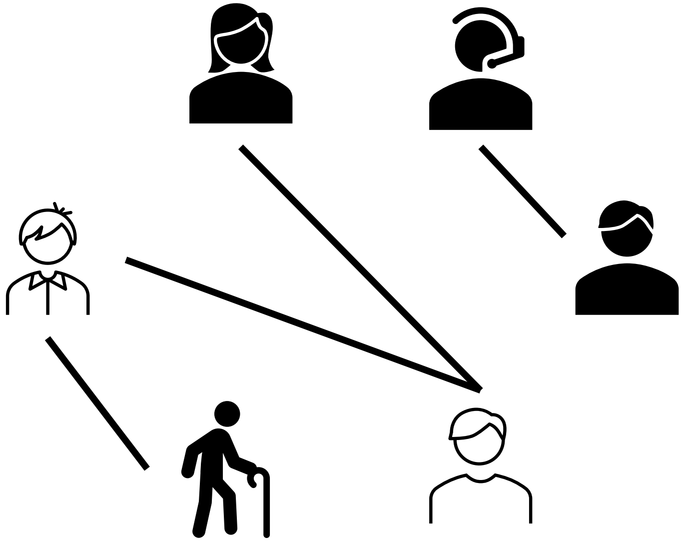

$\forall x (\exists y \ \texttt{AreFriends}(x, y))$

$\forall x \exists y \ \texttt{AreFriends}(x, y)$

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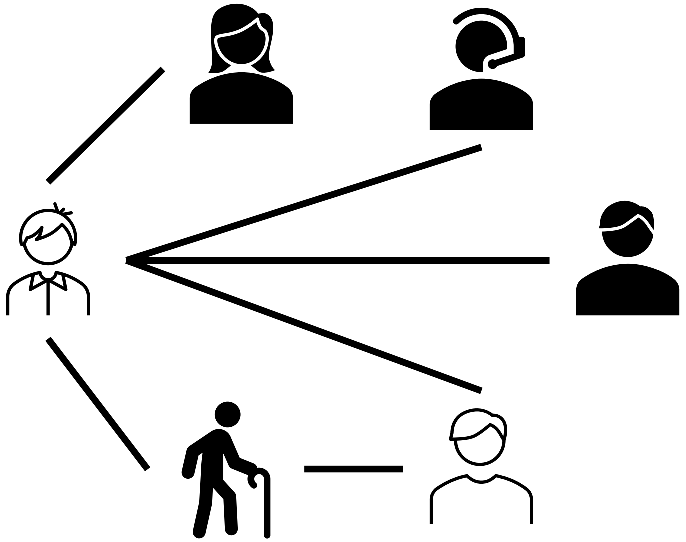


$\exists x (\forall y \ \texttt{AreFriends}(x, y))$

$\exists x \forall y \ \texttt{AreFriends}(x, y)$

Nested Quantifiers

$\forall x \exists y\, P(x, y)$

"For every x there exists a y such that $P(x, y)$ is true."

y might change depending on the x (people have different friends!).

$\exists x \forall y\, P(x, y)$

"There is an x such that for all $y, P(x, y)$ is true."

There's a special, magical x value so that $P(x, y)$ is true regardless of y.

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

$\forall x \exists y P(x, y)$

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

A row, where every entry is T

$\forall x \exists y P(x, y)$

In every row there must be a T

y (columns), x (rows)

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Keep everything in order

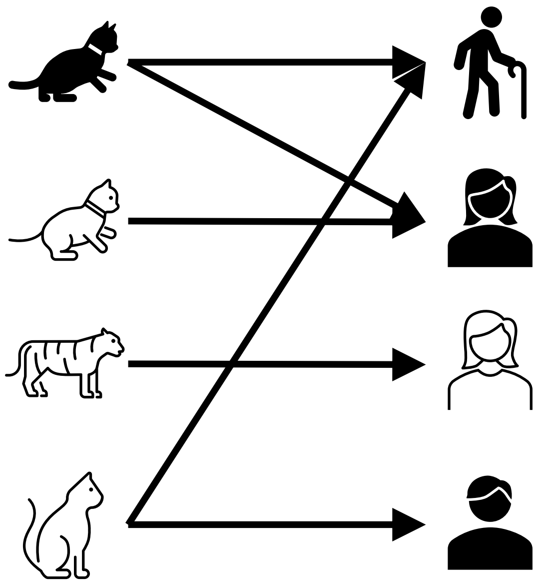
Keep the quantifiers in the same order in English as they are in the logical notation.

"There is someone out there for everyone" is a $\forall x \exists y$ statement in "everyday" English.

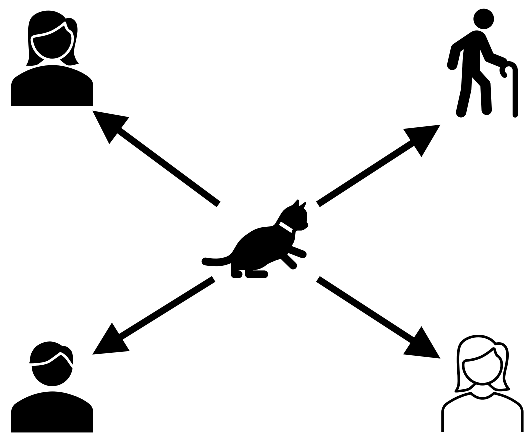
It would **never** be phrased that way in "mathematical English" We'll only every write "for every person, there is someone out there for them."

Try it yourselves

Every cat loves some human.



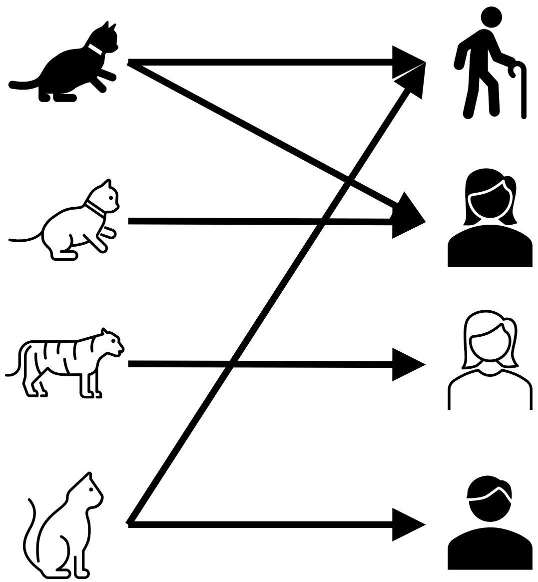
There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates `Cat`(x), `Dog`(x), and `Loves`(x, y) to mean x loves y.

Try it yourselves

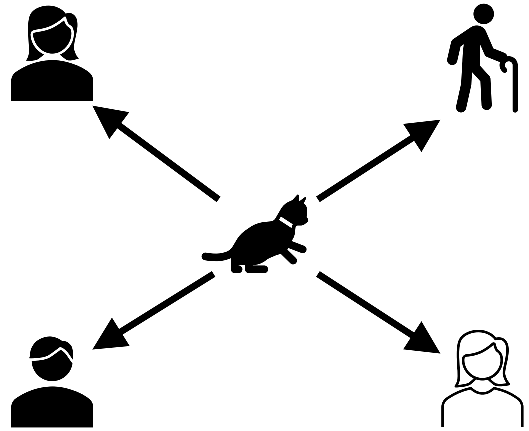
Every cat loves some human.



$\forall x\ (\mathtt{Cat}(x) \rightarrow \exists y[\mathtt{Human}(y) \wedge \mathtt{Loves}(x,y)])$

$\forall x \exists y(\mathtt{Cat}(x) \rightarrow [\mathtt{Human}(y) \wedge \mathtt{Loves}(x,y)])$

There is a cat that loves every human.



$\exists x\ (\mathtt{Cat}(x) \wedge \forall y[\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

$\exists x \forall y(\mathtt{Cat}(x) \wedge [\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

Negation

How do we negate nested quantifiers?

The old rule still applies.

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$$\neg(\forall x \exists y \forall z\, [P(x,y) \wedge Q(y,z)])$$

$$\exists x(\neg(\exists y \forall z\, [P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y(\neg(\forall z[P(x,y) \wedge Q(y,z)]))$$

$$\exists x \forall y \exists z(\neg[P(x,y) \wedge Q(y,z)])$$

$$\exists x \forall y \exists z[\neg P(x,y) \vee \neg Q(y,z)]$$

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y(\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y(\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x(\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$
(This statement is true, y can depend on x)

Theorems and Proofs

Theorems and Proofs

Theorem: A statement that has been proven to be true.

Proof: A valid argument that establishes a statement to be true.

You'll also see

"claim" (the statement we're about to prove)

"lemma" (small theorem, used to prove a bigger theorem)

"corollary" (small theorem, proven using a bigger theorem)

Theorems and Proofs

Examples of theorems include…

- Given a right triangle with side lengths a, b and hypotenuse c, $a^2 + b^2 = c^2$
- There are infinitely many prime numbers.
- There exists a problem that cannot be solved by a program.

Integer

We need a basic starting point to be able to prove things.

Objects to work with.

An integer: is any real number with no fractional part.

Some **definitions** to analyze

Even	Odd
`Even(x)` := An integer, x, is even if and only if there is an integer k such that $x = 2k$.	`Odd(x)` := An integer, x, is odd if and only if there is an integer k such that $x = 2k + 1$.

A word on definitions

Definitions are fundamental. Our goal is to communicate precisely.

When you come across an edge case, a definition is the way to solve it.

Is -4 even? Well $\exists k(-4 = 2k)$ (take $k = -2$), so yes it is!

We go to the definition. Not your gut feeling about what feels right.

How do we know something is true? Usually we verify the definition!

A word on definitions

How do we know something is true? Usually we verify the definition!

In other resources (textbooks, Wikipedia, etc.)

You will see things that look like this:

Definition: An integer, x, is even if $\exists k(x = 2k)$.

Notice it says "if" not "if and only if."

A definition is **always** an if and only if. The word "definition" has the "only if" direction in it.

I really wish people didn't do this. I wish they explicitly said "if and only if" but some people insist that "definition" implies the "only if" direction. Otherwise it's a "sufficient condition" not a "definition"

Proof Strategy: Direct Proof

Direct Proof

Direct proof is one strategy for proving statements of the form

$$\forall x[P(x) \to Q(x)]$$

Our First Direct Proof

Prove: "For all integers x, if x is even, then x^2 is even."

What's the claim in logic?

How would we prove this claim?

We'll see how to prove it formally in a minute; for now, just try to convince each other this statement is true.

Our First Direct Proof

Definitions

$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\mathrm{Even}(x) \rightarrow \mathrm{Even}\left(x^2\right)\right)$

Arbitrary

An "arbitrary" variable is one that

- is part of the domain of discourse (or some sub-domain you pick).
- You know **nothing** else about.

EVERY element of the domain could be plugged into that arbitrary variable. And everything else you say in the proof will follow.

An arbitrary variable is exactly what you need to convince us of a $\forall$.

If you want to prove a for-all you must explicitly tell us the variable is arbitrary.

- Your reader doesn't know what you're doing otherwise.

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\left(\text{Even}(x) \rightarrow \text{Even}\left(x^2\right)\right)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

Now What?

Definitions
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$

Well....what does it mean to be even?

$x = 2k$ for some integer k.

Where do we need to end up?

$\texttt{Even}(x^2)$

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

So x^2 is even.

Our First Direct Proof

Definitions

$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

So x^2 is even.

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

Squaring both sides, we see that:

$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$

Because k is an integer, $2k^2$ is also an integer.

So x^2 is two times an integer.

Which is exactly the definition of even, so x^2 is even.

Since x was an arbitrary integer, we conclude that for all integers x, if x is even then x^2 is also even.

Direct Proof Template

Prove: $\forall x \left(\mathrm{Even}(x) \rightarrow \mathrm{Even}(x^2)\right)$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Direct Proof Steps

These are the usual steps. We'll see different outlines in the future!!

- Introduction
  - Declare an arbitrary variable for each ∀ quantifier
  - Assume the left side of the implication
- Core of the proof
  - Unroll the predicate definitions
  - Manipulate towards the goal (using creativity, algebra, etc.)
  - Reroll definitions into the right side of the implication
- Conclude that you have proved the claim