you would expect that if team a beats team b and team b beats team c then team a should also beat team c.

This is not the case for Pac 12 football (in 2019)

parity.

2019 results by /u/ jensyao on reddit

Relations And Graphs

CSE 311 Autumn 2023
Lecture 23

Announcements

Midterm grades are out. Solutions are on Ed.

There's an Ed Post with how to interpret grades so far.

Remember you can schedule time to meet with TAs 1:1

- Can walkthrough your exam with you/give you tips on studying.
- Discuss previous concepts before the final exam.

Can also ask questions like these in regular office hours.

Can meet with Robbie 1:1 as well (see the Ed post)

- Those topics, but also discussions on grades.

Coming soon: an extra video with common midterm misconceptions.

Relations

Relations

A (binary) relation from A to B is a subset of $A \times B$
A (binary) relation on A is a subset of $A \times A$

Wait what?

$\leq$ is a relation on $\mathbb{Z}$.

"$3 \leq 4$" is a way of saying "3 relates to 4" (for the $\leq$ relation)

$(3,4)$ is an element of the set that defines the relation.

Properties of relations

What do we do with relations? Usually we prove properties about them.

Symmetry
A binary relation R on a set S is "symmetric" iff
for all $a, b \in S$, $[(a, b) \in R \rightarrow (\mathrm{b}, \mathrm{a}) \in R]$

$=$ on Σ^* is symmetric, for all $a, b \in \Sigma^*$ if $a = b$ then $b = a$.
$\subseteq$ is not symmetric on $\mathcal{P}(\mathcal{U})$ – $\{1,2,3\} \subseteq \{1,2,3,4\}$ but $\{1,2,3,4\} \nsubseteq \{1,2,3\}$

Transitivity
A binary relation R on a set S is "transitive" iff
for all $a, b, c \in S$, $[(a, b) \in R \land (b, c) \in R \rightarrow (\mathrm{a}, \mathrm{c}) \in R]$

$=$ on Σ^* is transitive, for all $a, b, c \in \Sigma^*$ if $a = b$ and $b = \mathrm{c}$ then $a = c$.
$\subseteq$ is transitive on $\mathcal{P}(\mathcal{U})$ – for any sets A, B, C if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
$\in$ is not a transitive relation – $1 \in \{1,2,3\}$, $\{1,2,3\} \in \mathcal{P}(\{1,2,3\})$ but $1 \notin \mathcal{P}(\{1,2,3\})$.

More Properties of relations

What do we do with relations? Usually we prove properties about them.

Antisymmetry
A binary relation R on a set S is "antisymmetric" iff
for all $a, b \in S$, $[(a, b) \in R \land a \neq b \rightarrow (\mathbf{b}, \mathbf{a}) \notin R]$

$\leq$ is antisymmetric on $\mathbb{Z}$

Reflexivity
A binary relation R on a set S is "reflexive" iff
for all $a \in S$, $[(a, a) \in R]$

$\leq$ is reflexive on $\mathbb{Z}$

$\leq$

You've proven antisymmetry too!

(a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Solution:

Suppose that $a \mid b$ and $b \mid a$, where a, b are integers. By the definition of divides, we have $a \neq 0$, $b \neq 0$ and $b = ka, a = jb$ for some integers k, j. Combining these equations, we see that $a = j(ka)$.

Then, dividing both sides by a, we get $1 = jk$. So, $\frac{1}{j} = k$. Note that j and k are integers, which is only possible if $j, k \in \{1, -1\}$. It follows that $b = -a$ or $b = a$.

Antisymmetry

A binary relation R on a set S is "antisymmetric" iff for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \rightarrow (\mathrm{b}, \mathrm{a}) \notin R]$

You showed | is antisymmetric on $\mathbb{Z}^+$ in section 5.

for all $a, b \in S$, $[(a, b) \in R \wedge (\mathrm{b}, \mathrm{a}) \in R \rightarrow a = b]$ is equivalent to the definition in the box above

The box version is easier to understand, the other version is usually easier to prove.

Try a few of your own

Decide whether each of these relations are

Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$ on $\mathcal{P}(\mathcal{U})$

$\geq$ on $\mathbb{Z}$

$>$ on $\mathbb{R}$

$|$ on $\mathbb{Z}^+$

$|$ on $\mathbb{Z}$

$\equiv (mod\ 3)$ on $\mathbb{Z}$

Symmetry: for all $a, b \in S$, $[(a, b) \in R \rightarrow (\mathrm{b}, \mathrm{a}) \in R]$

Antisymmetry: for all $a, b \in S$, $[(a, b) \in R \wedge a \neq b \rightarrow (\mathrm{b}, \mathrm{a}) \notin R]$

Transitivity: for all $a, b, c \in S$, $[(a, b) \in R \wedge (b, c) \in R \rightarrow (\mathrm{a}, \mathrm{c}) \in R]$

Reflexivity: for all $a \in S$, $[(a, a) \in R]$

Try a few of your own

Symmetry: for all $a, b \in S$, $[(a,b) \in R \to (\mathrm{b},\mathrm{a}) \in R]$

Antisymmetry: for all $a, b \in S$, $[(a,b) \in R \wedge a \neq b \to (\mathrm{b},\mathrm{a}) \notin R]$

Transitivity: for all $a, b, c \in S$, $[(a,b) \in R \wedge (b,c) \in R \to (\mathrm{a},\mathrm{c}) \in R]$

Reflexivity: for all $a \in S$, $[(a,a) \in R]$

Decide whether each of these relations are

Reflexive, symmetric, antisymmetric, and transitive.

$\subseteq$ on $\mathcal{P}(\mathcal{U})$ reflexive, antisymmetric, transitive

$\geq$ on $\mathbb{Z}$ reflexive, antisymmetric, transitive

$>$ on $\mathbb{R}$ antisymmetric, transitive

$|$ on $\mathbb{Z}^+$ reflexive, antisymmetric, transitive

$|$ on $\mathbb{Z}$ reflexive, transitive

$\equiv (mod\ 3)$ on $\mathbb{Z}$ reflexive, symmetric, transitive

How Do symmetry and antisymmetry relate?

There are relations that are neither symmetric nor antisymmetric.

For example $R = \{(1,2), (2,1), (1,3)\}$

$(1,2), (2,1)$ say you can't be antisymmetric.

$(1,3)$ [without $(3,1)$] says you can't be symmetric.

But you can only be both if the implications are vacuous.

A relation like $\{(1,1), (2,2), (3,3)\}$ is vacuously symmetric AND antisymmetric. Such relations are rarely seen though. Once you have x, y where $\mathrm{x} \neq y$ and $(x, y) \in \mathrm{R}$ the relation cannot be both.

Two Prototype Relations

A lot of fundamental relations follow one of two prototypes:

Equivalence Relation

A relation that is reflexive, symmetric, and transitive is called an "equivalence relation"

Partial Order Relation

A relation that is reflexive, antisymmetric, and transitive is called a "partial order"

Equivalence Relations

Equivalence relations "act kinda like equals"

$\equiv \pmod{n}$ is an equivalence relation.

$\equiv$ on compound propositions is an equivalence relation.

Fun fact: Equivalence relations "partition" their elements.

An equivalence relation R on S divides S into sets $S_1, \ldots S_k$ such that.

$\forall s\ (s \in S_i \text{ for some } i)$

$\forall s, s'$ $(s, s' \in S_i$ for some i if and only if $(s, s') \in R$)

$S_i \cap S_j = \emptyset$ for all $i \neq j$

Partial Orders

Partial Orders "behave kinda like less than or equal to"

In the sense that they put things in order

But it's only kinda like less than – it's possible that some elements can't be compared.

$|$ on $\mathbb{Z}^+$ is a partial order

$\subseteq$ on $\mathcal{P}(\mathcal{U})$ is a partial order

x is a prerequisite of (or-equal-to) y is a partial order on CSE courses

Why Bother?

If you prove facts about all equivalence relations or all partial orders, you instantly get facts in lots of different contexts.

If you learn to recognize partial orders or equivalence relations, you can get **a lot** of intuition for new concepts in a short amount of time.

Why now? We'll want relations over the next few weeks (and it's a convenient way to review proving implications, for all statements, and so on)

Graphs

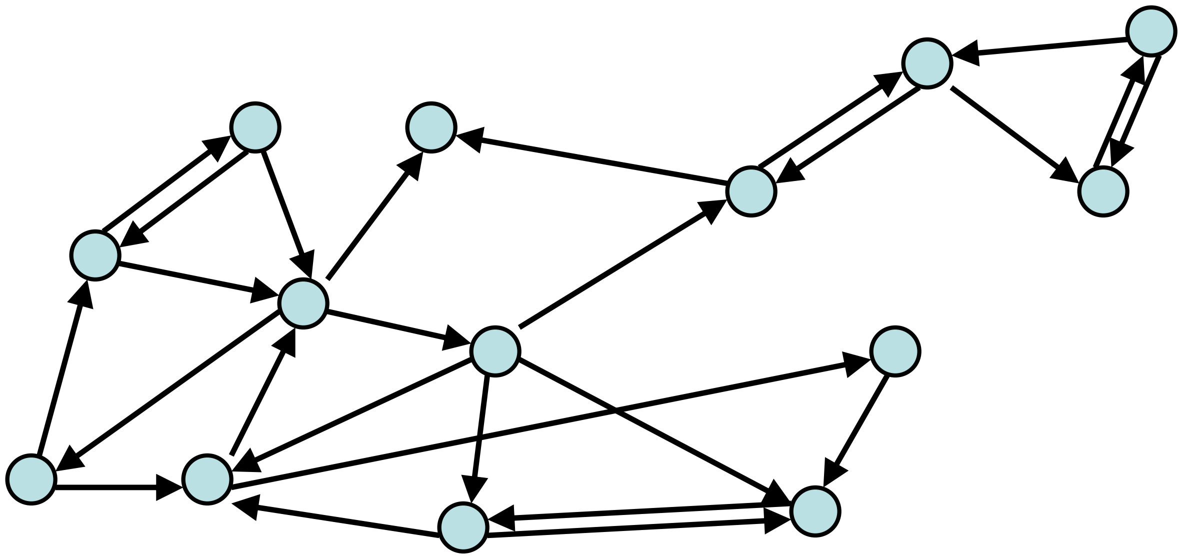
Directed Graphs

$G = (V, E)$

V is a set of vertices (an underlying set of elements)

E is a set of edges (ordered pairs of vertices; i.e. connections from one to the next).

Path $v_0, v_1, \dots, v_k$ such that $(v_i, v_{i+1}) \in E$
Simple Path: path with all v_i distinct
Cycle: path with $v_0 = v_k$ (and $k > 0$)
Simple Cycle: simple path plus edge (v_k, v_0) with $k > 0$



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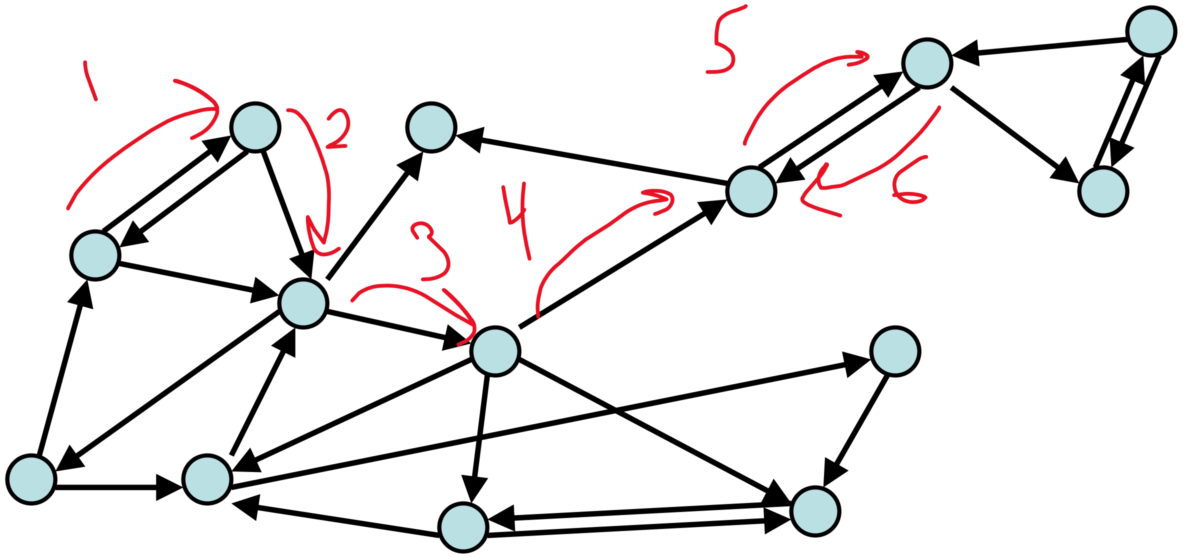

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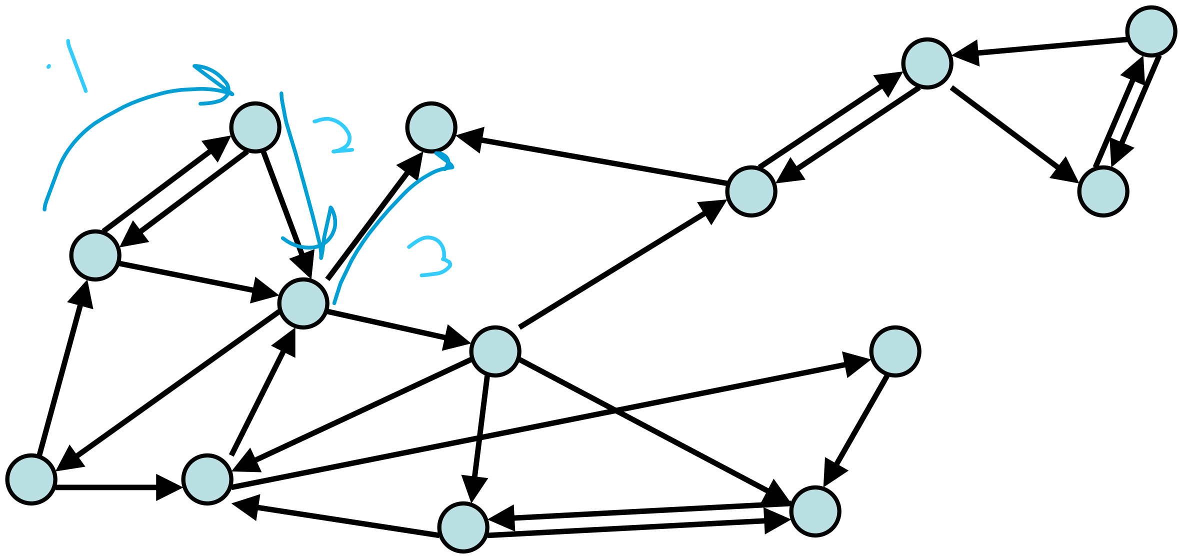

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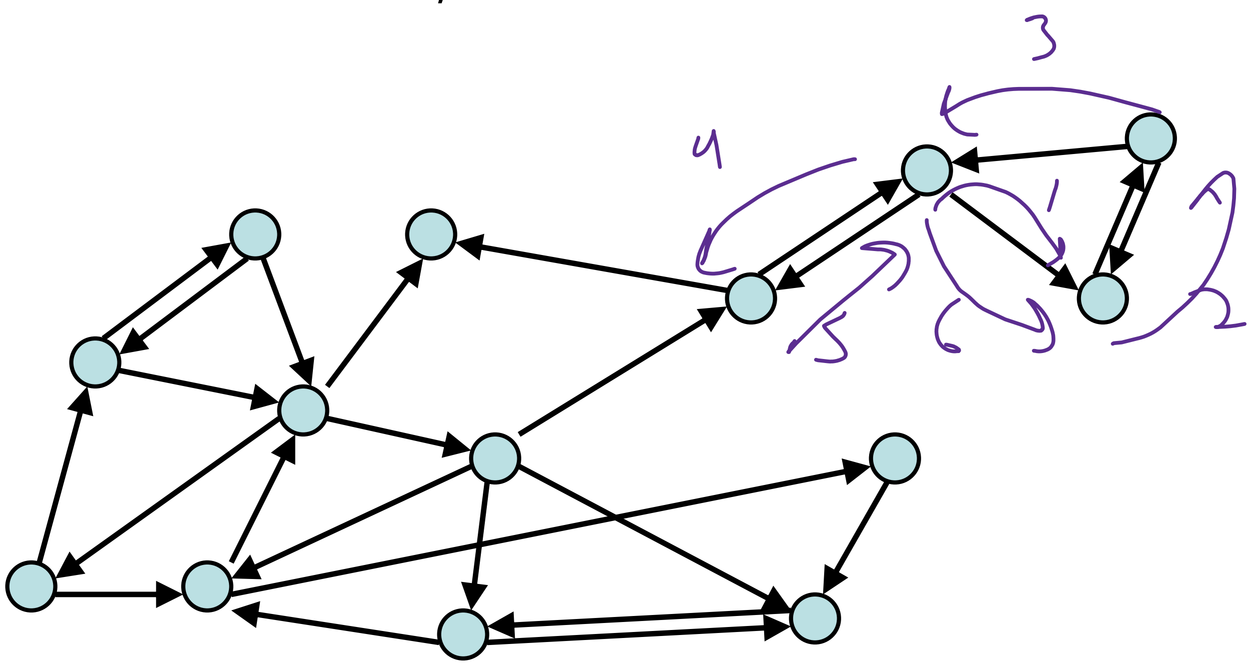

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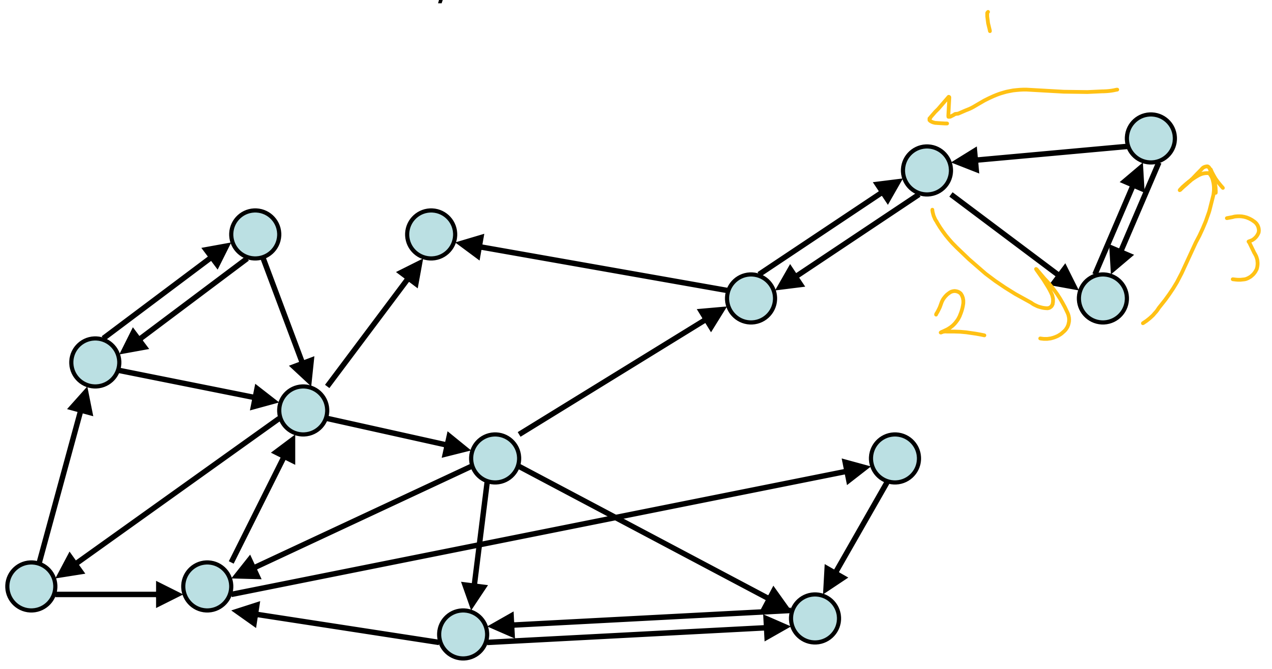


Lecture-Only Content

Relations and Graphs

More Relations and Graphs

The rest of this deck is a little more on:

Relations, specifically combining them together

Graphs, specifically representing relations as graphs.

We’re going to go through it ***very*** fast. We won’t have homework or exam questions on anything in this section of the deck.

But it is stuff you should see at least once because it might come back in future classes.

Combining Relations

Given a relation R from A to B

And a relation S from B to C,

The relation $S \circ R$ from A to C is

$\{(a, c) : \exists b[(a, b) \in R \wedge (b, c) \in S]\}$

Yes, I promise it's $S \circ R$ not $R \circ S$ – it makes more sense if you think about relations $(x, f(x))$ and $(x, g(x))$

But also don't spend a ton of energy worrying about the order, we almost always care about $R \circ R$, where order doesn't matter.

Combining Relations

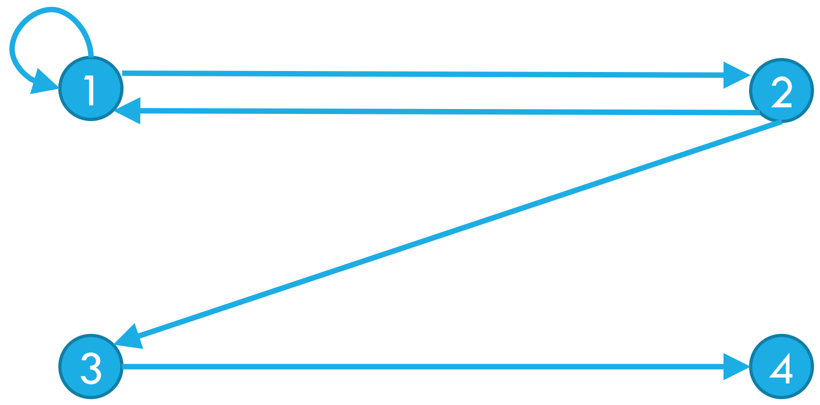
To combine relations, it’s a lot easier if we can see what’s happening.

We’ll use a representation of a directed graph

Representing Relations

To represent a relation R on a set A, have a vertex for each element of A and have an edge (a, b) for every pair in R.

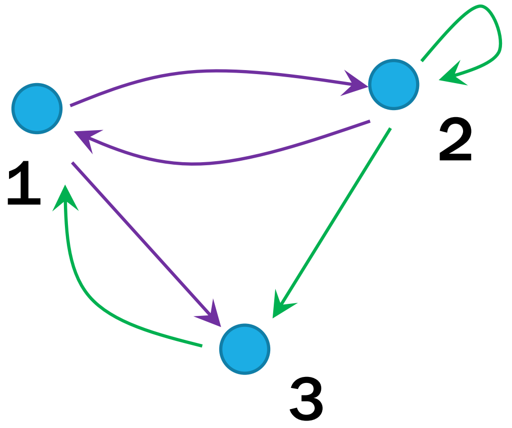
Let A be $\{1,2,3,4\}$ and R be $\{(1,1), (1,2), (2,1), (2,3), (3,4)\}$

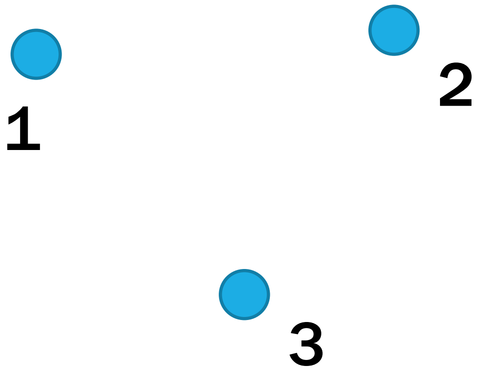


Combining Relations

If $\boldsymbol{S} = \{(2,2),(2,3),(3,1)\}$ and $\boldsymbol{R} = \{(1,2),(2,1),(1,3)\}$

Compute $\boldsymbol{S} \circ \boldsymbol{R}$ i.e. every pair (a,c) with a b with $(a,b) \in R$ and $(b,c) \in S$





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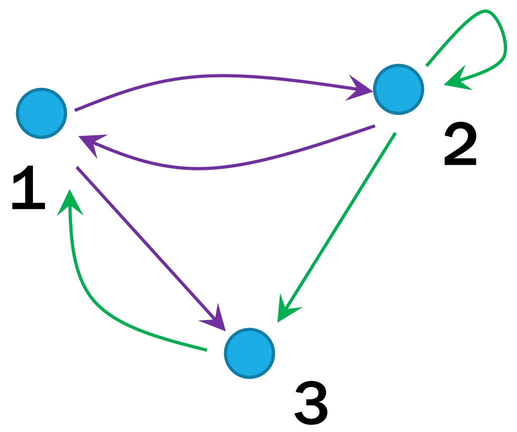

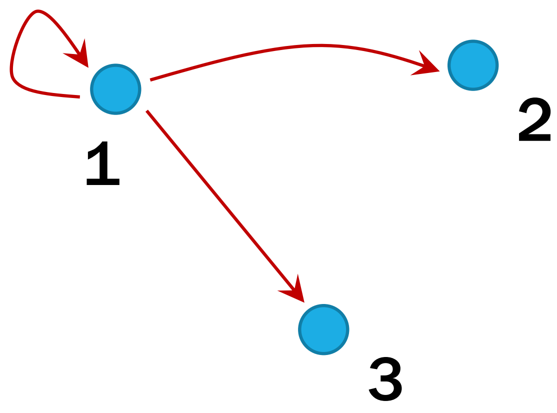

Combining Relations

Let R be a relation on A.

Define R^0 as $\{(a, a) : a \in A\}$

$R^k = R^{k-1} \circ R$

$(a, b) \in R^k$ if and only if there is a path of length k from a to b in R.

We can find that on the graph!

More Powers of R.

For two vertices in a graph, a can reach b if there is a path from a to b.

Let R be a relation on the set A. The connectivity relation R^* consists of all pairs (a, b) such that a can reach b (i.e. there is a path from a to b in R)

$R^* = \bigcup_{k=0}^{\infty} R^k$

Note we're starting from 0 (the textbook makes the unusual choice of starting from $k = 1$).

What’s the point of R^*

R^* is also the “reflexive-transitive closure of R.”

It answers the question “what’s the minimum amount of edges I would need to add to R to make it reflexive and transitive?”

Why care about that? The transitive-reflexive closure can be a summary of data – you might want to precompute it so you can easily check if a can reach b instead of recomputing it every time.

Relations and Graphs

Describe how each property will show up in the graph of a relation.

Reflexive

Symmetric

Antisymmetric

Transitive

Relations and Graphs

Describe how each property will show up in the graph of a relation.

Reflexive

Every vertex has a “self-loop” (an edge from the vertex to itself)

Symmetric

Every edge has its “reverse edge” (going the other way) also in the graph.

Antisymmetric

No edge has its “reverse edge” (going the other way) also in the graph.

Transitive

If there’s a length-2 path from a to b then there’s a direct edge from a to b