Section 04: Solutions

1. Formal Spoofs

For each of the following proofs, determine why the proof is incorrect. Then, consider whether the conclusion of the proof is true or not. If it is true, state how the proof could be fixed. If it is false, give a counterexample.

(a) Show that $\exists z\ \forall x\ P(x,z)$ follows from $\forall x\ \exists y\ P(x,y)$.

1.	$\forall x\ \exists y\ P(x,y)$	[Given]
2.	$\forall x\ P(x,c)$	[$\exists$ Elim: 1, c special]
3.	$\exists z\ \forall x P(x,z)$	[$\exists$ Intro: 2]

Solution:

The mistake is on line 2 where an inference rule is used on a subexpression. When we apply something like the $\exists$ Elim rule, the $\exists$ must be at the start of the expression and outside all other parts of the statement.

The conclusion is false, it's basically saying we can interchange the order of $\forall$ and $\exists$ quantifiers. Let the domain of discourse be integers and define $P(x,y)$ to be $x < y$. Then the hypothesis is true: for every integer, there is a larger integer. However, the conclusion is false: there is no integer that is larger than every other integer. Hence, there can be no correct proof that the conclusion follows from the hypothesis.

(b) Show that $\exists z\ (P(z) \wedge Q(z))$ follows from $\forall x\ P(x)$ and $\exists y\ Q(y)$.

1.	$\forall x\ P(x)$	[Given]
2.	$\exists y\ Q(y)$	[Given]
3.	Let z be arbitrary	
4.	$P(z)$	[$\forall$ Elim: 1]
5.	$Q(z)$	[$\exists$ Elim: 2, let z be the object that satisfies $Q(z)$]
6.	$P(z) \wedge Q(z)$	[$\wedge$ Intro: 4, 5]
7.	$\exists z\ P(z) \wedge Q(z)$	[$\exists$ Intro: 6]

Solution:

The mistake is on line 5. The $\exists$ Elim rule must create a new variable rather than applying some property to an existing variable.

The conclusion is true in this case. Instead of declaring z to be arbitrary and then applying $\exists$ Elim to make it specific, we can instead just apply the $\exists$ Elim rule directly to create z. To do this, we would remove lines 3 and 5 and define z by applying $\exists$ Elim to line 2. Note, it's important that we define z before applying line 4.

2. Just The Setup

For each of these statements,

- Translate the sentence into predicate logic.
- Write the first few and last few steps of an inference proof of the statement (you do not need to write the middle – just enough to introduce all givens and assumptions and the conclusion at the end)

- Write the first few sentences and last few sentences of the English proof.

(a) The product of an even integer and an odd integer is even.

Solution:

$\forall x \forall y([\mathsf{Even}(x) \land \mathsf{Odd}(y)] \rightarrow \mathsf{Even}(xy))$

1.Let a, b be arbitrary	–
2.1 $\mathsf{Even}(a) \land \mathsf{Odd}(b)$	[Assumption]
...	
2.? $\mathsf{Even}(ab)$	[?]
3.$[\mathsf{Even}(a) \land \mathsf{Odd}(b)] \rightarrow \mathsf{Even}(ab)$	[Direct Proof Rule]
4.$\forall y([\mathsf{Even}(a) \land \mathsf{Odd}(y)] \rightarrow \mathsf{Even}(ay))$	[Intro $\forall$]
5.$\forall x \forall y([\mathsf{Even}(x) \land \mathsf{Odd}(y)] \rightarrow \mathsf{Even}(xy))$	[Intro $\forall$]

Let x be an arbitrary even integer and let y be an arbitrary odd integer.
...
So xy is even.
Since x, y were arbitrary, we have that the product of an even integer with an odd integer is always even.

(b) There is an integer x s.t. $x^2 > 10$ and $3x$ is even.

Solution:

$\exists x[\mathsf{GreaterThan10}(x^2) \land \mathsf{Even}(3x)]$

...	
?.$\mathsf{GreaterThan10}(6^2)$	[Definition of $\mathsf{GreaterThan10}$]
...	
?.$\exists k[3 \cdot 6 = 2k]$	[?]
?.$\mathsf{Even}(3 \cdot 6)$	[Definition of Even]
?.$\mathsf{GreaterThan10}(6^2) \land \mathsf{Even}(3 \cdot 6)$	[Intro $\land$]
?.$\exists x[\mathsf{GreaterThan10}(x^2) \land \mathsf{Even}(3x)]$	[Intro $\exists$]

Consider $x = 6$.
...
Then there exists some integer k s.t. $3 \cdot 6 = 2k$.
So $6^2 > 10$ and $3 \cdot 6$ is even.
Hence, 6 is the desired x.

(c) For every integer n, there is a prime number p greater than n.

Solution:

$\forall x \exists y[\mathsf{Prime}(y) \land \mathsf{GreaterThan}(y, x)]$

1.Let a be an arbitrary object –
...
?. $\texttt{Prime}(b)$ [Definition of Prime]
...
?. $\texttt{GreaterThan}(b, a)$ [Definition of GreaterThan]
?. $\texttt{Prime}(b) \land \texttt{GreaterThan}(b, a)$ [Intro $\land$]
?. $\exists y[\texttt{Prime}(y) \land \texttt{GreaterThan}(y, a)]$ [Intro $\exists$]
?. $\forall x \exists y[\texttt{Prime}(y) \land \texttt{GreaterThan}(y, x)]$ [Intro $\forall$]

Let x be an arbitrary integer.
Consider $y = p$ (*this p is a specific prime*).
...
So p is prime and $p > x$.
Since x was arbitrary, we have that every integer has a prime number that is greater than it.

(d) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ for any sets A, B, C.

Solution:

$(A \subseteq B \land B \subseteq C) \to A \subseteq C$

1.1. $A \subseteq B \land B \subseteq C$ [Assumption]
1.2.Let a be an arbitrary object –
1.3.1. $a \in A$ [Assumption]
...
1.3.?. $a \in C$ [?]
1.3. $a \in A \to a \in C$ [Direct Proof Rule]
1.4. $\forall x[x \in A \to x \in C]$ [Intro $\forall$: 1.2, 1.3]
1.5 $A \subseteq C$ [Definition of $\subseteq$: 1.4]
1. $(A \subseteq B \land B \subseteq C) \to A \subseteq C$ [Direct Proof Rule]

Let A, B, C be arbitrary sets.
Suppose $A \subseteq B$ and $B \subseteq C$.
Let a be an arbitrary element of A.
...
Hence, a is an element of C.
Since a was arbitrary, every element of A is an element of C, so $A \subseteq C$.

3. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

Solution:

3

(b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \dots\}$

Solution:

$$\begin{aligned} B &= \{\{\},\ \{\{\}\},\ \{\{\},\{\}\},\ \{\{\},\{\},\{\}\},\ \ldots\} \\ &= \{\{\},\ \{\{\}\},\ \{\{\}\},\ \{\{\}\},\ \ldots\} \\ &= \{\varnothing, \{\varnothing\}\} \end{aligned}$$

So, there are two elements in B.

(c) $C = A \times (B \cup \{7\})$

Solution:

$C = \{1, 2, 3\} \times \{\varnothing, \{\varnothing\}, 7\} = \{(a, b) \mid a \in \{1, 2, 3\}, b \in \{\varnothing, \{\varnothing\}, 7\}\}$. It follows that there are $3 \times 3 = 9$ elements in C.

(d) $D = \varnothing$

Solution:

0.

(e) $E = \{\varnothing\}$

Solution:

1.

(f) $F = \mathcal{P}(\{\varnothing\})$

Solution:

$2^1 = 2$. The elements are $F = \{\varnothing, \{\varnothing\}\}$.

4. Set = Set

Prove the following set identities. Write both a formal inference proof **and** an English proof.

(a) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \backslash B$ for any sets A, B.

Solution:

1.	Let x be an arbitrary object		
	2.1 $x \in A \cap \overline{B}$	Assumption	
	2.2 $x \in A \wedge x \in \overline{B}$	Def of $\cap$	
	2.3 $x \in A \wedge x \notin B$	Def of $\overline{B}$	
	2.4 $x \in A \backslash B$	Def of $\backslash$	
2.	$x \in A \cap \overline{B} \rightarrow x \in A \backslash B$		Direct Proof
3.	$\forall x(x \in A \cap \overline{B} \rightarrow x \in A \backslash B)$		Intro $\forall$
4.	$A \cap \overline{B} \subseteq A \backslash B$		Def of $\subseteq$

Let x be an arbitrary element and suppose that $x \in A \cap \overline{B}$. By definition of intersection, $x \in A$ and $x \in \overline{B}$, so by definition of complement, $x \notin B$. Then, by definition of set difference, $x \in A \backslash B$. Since x was arbitrary, we can conclude that $A \cap \overline{B} \subseteq A \backslash B$ by definition of subset.

(b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

Solution:

1.	Let x be arbitrary		
	2.1 $x \in (A \cap B) \times C$	Assumption	
	2.2 $(y, z) \in (A \cap B) \times C$	Def of $\times$	
	2.3 $y \in (A \cap B) \wedge z \in C$	Def of $\times$	
	2.4 $y \in (A \cap B)$	Elim $\wedge$	
	2.5 $y \in A \wedge y \in B$	Def of $\cap$	
	2.6 $y \in A$	Elim $\wedge$	
	2.7 $z \in C$	Elim $\wedge$	
	2.8 $z \in C \vee z \in D$	Intro $\vee$	
	2.9 $z \in (C \cup D)$	Def of $\cup$	
	2.10 $y \in A \wedge z \in (C \cup D)$	Intro $\wedge$	
	2.11 $(y, z) \in A \times (C \cup D)$	Def of $\times$	
	2.12 $x \in A \times (C \cup D)$	Def of $\times$	
2.	$x \in (A \cap B) \times C \rightarrow x \in A \times (C \cup D)$		Direct Proof
3.	$\forall x(x \in (A \cap B) \times C \rightarrow x \in A \times (C \cup D))$		Intro $\forall$
4.	$(A \cap B) \times C \subseteq A \times (C \cup D)$		Def of $\subseteq$

Let x be an arbitrary element of $(A \cap B) \times C$. Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$. Since $y \in A \cap B$, $y \in A$ and $y \in B$ by definition of $\cap$; in particular, all we care about is that $y \in A$. Since $z \in C$, by definition of $\cup$, we also have $z \in C \cup D$. Therefore since $y \in A$ and $z \in C \cup D$, by definition of Cartesian product we have $x = (y, z) \in A \times (C \cup D)$.

Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$ as required.

5. Set Equality

(a) Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Solution:

Let x be an arbitrary member of $A \cap (A \cup B)$. Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$. Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

Now let y be an arbitrary member of A. Then $y \in A$. So certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in A \cup B$. Since $y \in A$ and $y \in A \cup B$, by definition of intersection, $y \in A \cap (A \cup B)$. Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

Therefore $A \cap (A \cup B) = A$, by containment in both directions.

(b) Let $\mathcal{U}$ be the universal set. Show that $\overline{\overline{X}} = X$.

Solution:

Let x be arbitrary. Suppose x is an element of $\overline{\overline{X}}$. By definition of complement, $x \in \mathcal{U} \setminus \overline{X}$, or equivalently $x \notin \overline{X}$. Applying the definition of complement again, we have $x \notin (\mathcal{U} \setminus X)$, which we can write $\neg(x \in \mathcal{U} \setminus X)$; that is, $\neg(x \notin X)$. By the double negation law we have $x \in X$. Since x was arbitrary, we have shown that $\overline{\overline{X}} \subseteq X$.

Now let y be arbitrary and suppose y is an element of X. Applying the double-negation law, we have $\neg\neg(y \in X)$, which is $\neg(y \notin X)$. Since $y \in \mathcal{U}$, $\neg(y \in \mathcal{U} \wedge x \notin X)$, which is equivalent to $\neg(y \in \mathcal{U} \setminus X)$. Using the definition of $\notin$, we can write $y \notin \mathcal{U} \setminus X$. By definition of $\mathcal{U}$, we have $y \in \mathcal{U} \setminus (\mathcal{U} \setminus X)$. Applying the definition of complement, we get $y \in \mathcal{U} \setminus \overline{X}$. Applying the definition of complement again, we have $y \in \overline{\overline{X}}$. Hence, since y was arbitrary, $X \subseteq \overline{\overline{X}}$.

Therefore, $\overline{\overline{X}} = X$ (by mutual containment).

6. Trickier Set Theory

Note, this problem requires a little more thinking. The solution will cover both the answer as well as the intuition used to arrive at it.

Show that for any set X and any set A such that $A \in \mathcal{P}(X)$, there exists a set $B \in \mathcal{P}(X)$ such that $A \cap B = \emptyset$ and $A \cup B = X$.

Solution:

This solution might look long, but most of it is explaining the intuition. The proof itself is fairly short!

We start by letting X and A be arbitrary sets and assume that $A \in \mathcal{P}(X)$. Now we think about our goal. We want to show there is some set B with the given properties. The way to do this is usually to construct B somehow, but there's nothing in the problem that tells us where B might come from!

When you get stuck like this, try to use all the information given in the problem to deduce as many things as we can. First we might notice that $A \in \mathcal{P}(X)$ means that $A \subseteq X$ and $B \in \mathcal{B}$ means $B \subseteq X$. So given some subset of X, we must construct some other subset.

Next, we consider what we know about B. The property that $A \cap B = \emptyset$ means that B and A share no elements in common. That is, B consists only of elements in X that are not in A. The property that $A \cup B = X$ is a little tricker. We might think of A as some collection of objects from X, $A \cup B$ throws in all the elements of B, and once we do that we have all the elements of X. In order for this to happen, we know B must contain all the elements of X that weren't in A.

At this point we've deduced that B contains only elements in X that are not in A, but also that it must contain all the elements of X that are not in A. This says that B is exactly the elements of X that are not in A. Does this sound familiar? It's exactly the set difference $X \setminus A$.

Now we can write out the proof. Let X be an arbitrary set and let A be an arbitrary element of $\mathcal{P}(X)$. Let

$B = X \setminus A$. For any $x \in X \setminus A$, by definition we have $x \in X$ which shows that $B \subseteq X$ and by definition $B \in \mathcal{P}(X)$.

To show that $A \cap B = \emptyset$, we must show that there are no elements that are both in A and B. If x is in $X \setminus A$, then by definition x is not in A, so there's no element that can be in both. Thus, $A \cap B = \emptyset$. To prove $A \cup B = X$, we first suppose $x \in A \cup B$ which by definition means $x \in A$ or $x \in B$. If $x \in A$ then since $A \subseteq X$ we have $x \in A$. If $x \in B$ then $x \in X \setminus A$ which by definition means that $x \in X$. In either case $x \in X$. In the other direction suppose $x \in X$. We again consider two cases. If $x \in A$ then there's nothing to show because then $x \in A \cup B$ automatically. If $x \notin A$ then since x is an element of X not in A, by definition we have $x \in X \setminus A$ which is equal to B, so in this case we also have $x \in A \cup B$. In either case $x \in A \cup B$. Since we've shown $x \in A \cup B$ if and only if $x \in X$, we've shown $A \cup B = X$, which completes the proof.

7. Predicate Logic Formal Proof

Given $\forall x. T(x) \to M(x)$, we wish to prove $(\exists x. T(x)) \to (\exists y. M(y))$. The following formal proof does this, but it is missing citations for which rules are used, and which lines they are based on. Fill in the blanks with inference rules or predicate logic equivalences, as well as the line numbers.

Then, summarize in English what is going on here.

1. $\forall x. T(x) \to M(x)$ (________________)
   - 2.1. $\exists x. T(x)$ (________________)

     Let r be the object that satisfies $T(r)$
   - 2.2. $T(r)$ (________________, from ______)
   - 2.3. $T(r) \to M(r)$ (________________, from ______)
   - 2.4. $M(r)$ (________________, from ______)
   - 2.5. $\exists y. M(y)$ (________________, from ______)
2. $(\exists x. T(x)) \to (\exists y. M(y))$ (________________, from ______)

Solution:

1. $\forall x. T(x) \to M(x)$ (Given)
   - 2.1. $\exists x. T(x)$ (Assumption)

     Let r be the object that satisfies $T(r)$
   - 2.2. $T(r)$ ($\exists$ elimination, from 2.1)
   - 2.3. $T(r) \to M(r)$ ($\forall$ elimination, from 1)
   - 2.4. $M(r)$ (Modus Ponens, from 2.2 and 2.3)
   - 2.5. $\exists y. M(y)$ ($\exists$ introduction, from 2.4)
2. $(\exists x. T(x)) \to (\exists y. M(y))$ (Direct Proof Rule, from 2.1-2.5)

Following the premise of the implication, we suppose there is an object that satisfies $T(\cdot)$. Then it must satisfy $M(\cdot)$ also, by the given, which gives us the conclusion of the implication.

8. Formal Proof (Direct Proof Rule)

Show that $\neg t \to s$ follows from $t \vee q$, $q \to r$ and $r \to s$. **Solution:**

1.	$t \vee q$		[Given]
2.	$q \to r$		[Given]
3.	$r \to s$		[Given]
	4.1. $\neg t$	[Assumption]	
	4.2. q	[Elim of $\vee$: 1, 4.1]	
	4.3. r	[MP of 4.2, 2]	
	4.4. s	[MP 4.3, 3]	
4.	$\neg t \to s$		[Direct Proof Rule]

9. Find the Bug

Each of these inference proofs is incorrect. Identify the line (or lines) which incorrectly apply a law, and explain the error. Then, if the claim is false, give concrete examples of propositions to show it is false. If it is true, write a correct proof.

(a) This proof claims to show that given $a \to (b \vee c)$, we can conclude $a \to c$.

1.	$a \to (b \vee c)$	[Given]
	2.1. a	[Assumption]
	2.2. $\neg b$	[Assumption]
	2.3. $b \vee c$	[Modus Ponens, from 1 and 2.1]
	2.4. c	[$\vee$ elimination, from 2.2 and 2.3]
2.	$a \to c$	[Direct Proof Rule, from 2.1-2.4]

Solution:

The error here is in lines 2.2 and 2. When beginning a subproof for the direct proof rule, only one assumption may be introduced. If the author of this proof wanted to assume both a and $\neg b$, they should have used the assumption $a \wedge \neg b$, which would make line 3 be $(a \wedge \neg b) \to c$ instead.

And the claim is false in general. Consider:
a: "I am outside"
b: "I am walking my dog"
c: "I am swimming"
If we assert "If I am outside, I am walking my dog or swimming," we cannot reasonably conclude that "If I am outside, I am swimming" ($a \to c$).

(b) This proof claims to show that given $p \to q$ and r, we can conclude $p \to (q \vee r)$.

1.$p \to q$	[Given]
2.r	[Given]
3.$p \to (q \vee r)$	[Intro $\vee$ (1,2)]

Solution:

Bug is in step 3, we're applying the rule to only a subexpression.

The statement is true though. A correct proof introduces p as an assumption, uses MP to get q, introduces $\vee$ to get $q \vee r$, and the direct proof rule to complete the argument.

(c) This proof claims to show that given $p \to q$ and q that we can conclude p

$1.p \to q$	[Given]
$2.q$	[Given]
$3.\neg p \vee q$	[Law of Implication (1)]
$4.p$	[Eliminate $\vee$ (2,3)

Solution:

The bug is in step 4. Eliminate $\vee$ from 3 would let us conclude $\neg p$ if we had $\neg q$ or q if we had p. It doesn't tell us anything with knowing q.

Indeed, the claim is false. We could have p: "it is raining"
q: "I have my umbrella"

And be a person who always carries their umbella with them (even on sunny days). The information is not sufficient to conclude p.

10. A Formal Proof in Predicate Logic

Prove $\exists x\ (P(x) \vee R(x))$ from $\forall x\ (P(x) \vee Q(x))$ and $\forall y\ (\neg Q(y) \vee R(y))$. **Solution:**

1.	$\forall x\ (P(x) \vee Q(x))$	[Given]
2.	$\forall y\ (\neg Q(y) \vee R(y))$	[Given]
3.	$P(a) \vee Q(a)$	[Elim $\forall$: 1]
4.	$\neg Q(a) \vee R(a)$	[Elim $\forall$: 2]
5.	$Q(a) \to R(a)$	[Law of Implication: 4]
6.	$\neg\neg P(a) \vee Q(a)$	[Double Negation: 3]
7.	$\neg P(a) \to Q(a)$	[Law of Implication: 5]
	8.1. $\neg P(a)$ [Assumption]	
	8.2. $Q(a)$ [Modus Ponens: 8.1, 7]	
	8.3. $R(a)$ [Modus Ponens: 8.2, 5]	
8.	$\neg P(a) \to R(a)$	[Direct Proof]
9.	$\neg\neg P(a) \vee R(a)$	[Law of Implication: 8]
10.	$P(a) \vee R(a)$	[Double Negation: 9]
11.	$\exists x\ (P(x) \vee R(x))$	[Intro $\exists$: 10]