CSE 311 Section 6

Induction

Administrivia

Announcements & Reminders

- HW4
  - Grades out now
  - If you think something was graded incorrectly, submit a regrade request!

- HW5 (BOTH PARTS)
  - BOTH PARTS due Wednesday 10/8 @ 10pm

- Midterm is Coming!!!
  - Wednesday 10/15 @ 6-7:30 pm in BAG 131 and 154
  - If you cannot make it, please let us know ASAP and we will schedule you for a makeup

Induction

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all n by induction on n.

<u>Base Case:</u> Show $P(b)$ is true.

<u>Inductive Hypothesis:</u> Suppose $P(k)$ holds for an arbitrary $k \geq b$.

<u>Inductive Step:</u> Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

<u>Conclusion:</u> Therefore, $P(n)$ holds for all n by the principle of induction.

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds **for all n** by induction on n.

Note: often you will condition n here, like "all natural numbers n" or "$n \geq 0$"

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k + 1)$ (i.e. get $P(k) \rightarrow P(k + 1)$)

Conclusion: Therefore, $P(n)$ holds **for all n** by the principle of induction.

Match the earlier condition on n in your conclusion!

Problem 1 – Induction with Equality

a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Lets walk through part (a) together.

We can “fill in” our induction template to construct our proof by induction.

Problem 1 – Induction with Equality

Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
for all $n \in \mathbb{N}$.

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

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Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

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Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0+1+2+\cdots+k=\frac{k(k+1)}{2}$

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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

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<u>Inductive Hypothesis:</u> Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

<u>Inductive Step:</u> Goal: Show $P(k + 1)$: $0 + 1 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

$0 + 1 + \cdots + k + (k + 1) = \cdots$

$\cdots$

$= \frac{(k+1)(k+2)}{2}$

?

<u>Conclusion:</u> Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

$$0 + 1 + \cdots + k + (k+1) = (0 + 1 + \cdots + k) + (k+1)$$

$$\ldots$$

$$= \frac{(k+1)(k+2)}{2} \qquad ?$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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$$
\begin{aligned}
0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&\dots \\
&= \frac{(k+1)(k+2)}{2} && ?
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

Show using induction that $0+1+2+\cdots+n=\frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0+1+2+\cdots+n=\frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

<u>Base Case:</u> $P(0)$: $0+\cdots=0=\frac{0(0+1)}{2}$ so the base case holds.

<u>Inductive Hypothesis:</u> Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0+1+2+\cdots+k=\frac{k(k+1)}{2}$

<u>Inductive Step:</u> Goal: Show $P(k+1)$: $0+1+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}$

$$
\begin{aligned}
0+1+\cdots+k+(k+1) &= (0+1+\cdots+k)+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) && \text{by I.H.} \\
&= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\
&\ldots \\
&= \frac{(k+1)(k+2)}{2} && ?
\end{aligned}
$$

<u>Conclusion:</u> Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

$$\begin{aligned} 0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} && ? \end{aligned}$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

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&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} && \text{factoring out (k+1)}
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Now try part (b) with people around you, and then we'll go over it together!

Problem 1 – Induction with Equality

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}$.

$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

Strong Induction

Why Strong Induction?

In **weak induction**, the inductive hypothesis only assumes that $P(k)$ is true and uses that in the inductive step to prove the implication $P(k) \rightarrow P(k+1)$.

In **strong induction**, the inductive hypothesis assumes the predicate holds for every step from the base case(s) up to $P(k)$. This usually looks something like $P(b_1) \wedge P(b_2) \wedge \cdots \wedge P(k)$. Then it uses this stronger inductive hypothesis in the inductive step to prove the implication $P(b_1) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$.

Strong induction is necessary when we have multiple base cases, or when we need to go back to a smaller number than k in our inductive step.

Strong Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all $n \geq b_{min}$ by induction on n.

Base Case: Show $P(b_{min}), P(b_{min+1}), \dots, P(b_{max})$ are all true.

Inductive Hypothesis: Suppose $P(b_{min}) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq b_{max}$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(b_{min}) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds for all $n \geq b_{min}$ by the principle of induction.

Problem 4 – Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

First, let's construct a formula for $f(n)$. How many rabbits does he have each year? Let's do some calculations, and see if we can find a pattern. Then, we'll use induction to prove the pattern holds for all n!

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Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = 2f(2-1) - f(2-2) = 2f(1) - f(0) = 2(1) - 0 = 2 - 0 = 2$$
$$f(3) = 2f(3-1) - f(3-2) = 2f(2) - f(1) = 2(2) - 1 = 4 - 1 = 3$$
$$f(4) = 2f(4-1) - f(4-2) = 2f(3) - f(2) = 2(3) - 2 = 6 - 2 = 4$$

It seems like we have a pattern here!

$f(n) = n$

But we don't want to have to check for EVERY n, so let's see if we can prove it with induction instead!

Problem 4 – Cantelli's Rabbits

What kind of induction should we use?

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Strong induction!

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What kind of induction should we use?

Strong induction!

Two big clues:

- Multiple base cases in the formula: $f(0) = 0$ and $f(1) = 1$
- Recursively defined step of formula goes back further than just n:
  - $f(n)$ based on both $f(n-1)$ and $f(n-2)$
  - for $P(n)$ to be true, both $P(n-1)$ and $P(n-2)$ must be true

Problem 4 – Cantelli's Rabbits

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all $n \geq b_{min}$ by induction on n.

Base Case: Show $P(b_{min}), P(b_{min+1}), \ldots, P(b_{max})$ are all true.

Inductive Hypothesis: Suppose $P(b_{min}) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq b_{max}$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(b_{min}) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds for all $n \geq b_{min}$ by the principle of induction.

Fill in the strong induction template to prove the claim!

Problem 4 – Cantelli's Rabbits

Let $P(n)$ be "".

We show $P(n)$ holds …

<u>Base Cases:</u>

<u>Inductive Hypothesis:</u>

<u>Inductive Step:</u>

<u>Conclusion:</u> Therefore, $P(n)$ holds for all … by the principle of induction.

That’s All, Folks!

Thanks for coming to section this week!
Any questions?