Section MR: Solutions

1. Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- $\texttt{soy}(x)$ is true iff x contains soy milk.
- $\texttt{whole}(x)$ is true iff x contains whole milk.
- $\texttt{sugar}(x)$ is true iff x contains sugar
- $\texttt{decaf}(x)$ is true iff x is not caffeinated.
- $\texttt{vegan}(x)$ is true iff x is vegan.
- $\texttt{RobbieLikes}(x)$ is true iff Robbie likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

(a) Coffee drinks with whole milk are not vegan. **Solution:**

$\forall x(\texttt{whole}(x) \to \neg\,\texttt{vegan}(x))$.

(b) Robbie only likes one coffee drink, and that drink is not vegan. **Solution:**

$\exists x \forall y(\texttt{RobbieLikes}(x) \land \neg\,\texttt{Vegan}(x) \land [\texttt{RobbieLikes}(y) \to x = y])$

OR $\exists x(\texttt{RobbieLikes}(x) \land \neg\,\texttt{Vegan}(x) \land \forall y[\texttt{RobbieLikes}(y) \to x = y])$

(c) There is a drink that has both sugar and soy milk. **Solution:**

$\exists x(\texttt{sugar}(x) \land \texttt{soy}(x))$

Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\texttt{decaf}(x) \land \texttt{RobbieLikes}(x)] \to \texttt{sugar}(x))$$

Solution:

Every decaf drink that Robbie likes has sugar.

Statements like "For every decaf drink, if Robbie likes it then it has sugar" are equivalent, but only partially take advantage of domain restriction.

2. Midterm Review: Set Theory

Suppose that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Solution:

Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be an arbitrary element. Then by definition of powerset, $X \subseteq A$. Let $y \in X$ be arbitrary. Then since $X \subseteq A$, by definition of subset, $y \in A$. Since $A \subseteq B$, by definition of subset again, $y \in B$. Since y was arbitrary in X, by definition of subset once more, $X \subseteq B$. Then by definition of powerset, $X \in \mathcal{P}(B)$. Since X was arbitrary in $\mathcal{P}(A)$, we have shown $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

3. Midterm Review: Number Theory

Let p be a prime number at least 3, and let x be an integer such that $x^2 \% p = 1$.

(a) Show that if an integer y satisfies $y \equiv 1 \pmod p$, then $y^2 \equiv 1 \pmod p$. (this proof will be short!) (Try to do this without using the theorem "Raising Congruences To A Power")

Solution:

Let y be an arbitrary integer and suppose $y \equiv 1 \pmod p$. We can multiply congruences, so multiplying this congruence by itself we get $y^2 \equiv 1^2 \pmod p$. Since y is arbitrary, the claim holds.

(b) Repeat part (a), but don't use any theorems from the Number Theory Reference Sheet. That is, show the claim directly from the definitions.

Solution:

Let x be an arbitrary integer and suppose $x \equiv 1 \pmod p$. By the definition of Congruences, $p \mid (x-1)$. Therefore, by the definition of divides, there exists an integer k such that

$$pk = (x-1)$$

By multiplying both sides of pk = (x - 1) by (x + 1) and re-arranging the equation, we have

$$pk(x+1) = (x-1)(x+1)$$
$$p(k(x+1)) = (x-1)(x+1)$$

Since $(x-1)(x+1) = x^2 - 1$, by replacing $(x-1)(x+1)$ with $x^2 - 1$, we have

$$p(k(x+1)) = x^2 - 1$$

Note that since k and x are integers, (k (x + 1)) is also an integer. Therefore, by the definition of divides $p \mid x^2 - 1$.
Hence, by the definition of Congruences, $x^2 \equiv 1 \pmod p$.

(c) From part (a), we can see that $x\%p$ can equal 1. Show that for any integer x, if $x^2 \equiv 1 \pmod p$, then $x \equiv 1 \pmod p$ or $x \equiv -1 \pmod p$. That is, show that the only value $x\%p$ can take other than 1 is $p-1$.
Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod p$ and use the fact that $x^2 - 1 = (x-1)(x+1)$
Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

Solution:

Let x be an arbitrary integer and suppose $x^2 \equiv 1 \pmod p$. By the definition of Congruences,

$$p \mid x^2 - 1$$

Since $(x-1)(x+1) = x^2 - 1$, by replacing $x^2 - 1$ with $(x-1)(x+1)$, we have

$$p \mid (x-1)(x+1)$$

Note that for an integer p if p is a prime number and $p \mid (ab)$, then $p \mid a$ or $p \mid b$. In this case, since p is a prime number, by applying the rule, we have $p \mid (x-1)$ or $p \mid (x+1)$.
Therefore, by the definition of Congruences, we have $x \equiv 1 \pmod p$ or $x \equiv -1 \pmod p$.

4. Midterm Review: Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Solution:

Let $\mathsf{P}(n)$ be the statement "$S_n = \frac{1}{6}n(n+1)(2n+1)$" defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case: When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $\mathsf{P}(0)$ is true.

Inductive Hypothesis: Suppose that $\mathsf{P}(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step: Examining S_{k+1}, we see that

$$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the inductive hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned}
S_{k+1} &= S_k + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right) \\
&= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$

Thus, we can conclude that $\mathsf{P}(k+1)$ is true.

Conclusion: $P(n)$ holds for all integers $n \geq 0$ by the principle of induction.

5. Midterm Review: Strong Induction

Robbie is planning to buy snacks for the members of his competitive roller-skating troupe. However, his local grocery store sells snacks in packs of 5 and packs of 7.

Prove that Robbie can buy exactly n snacks for all integers $n \geq 24$

Solution:

Let $\mathsf{P}(n)$ be the statement "Robbie can buy n snacks with packs of 5 and packs of 7 snacks" defined for all $n \geq 24$. We prove that $\mathsf{P}(n)$ is true for all $n \geq 24$ by the principle of strong induction.

Base Case:

n = 24: 24 snacks can be bought with 2 packs of 7 and 2 packs of 5 snacks.
n = 25: 25 snacks can be bought with 5 packs of 5 snacks.
n = 26: 26 snacks can be bought with 3 packs of 7 and 1 pack of 5 snacks.
n = 27: 27 snacks can be bought with 1 pack of 7 and 4 packs of 5 snacks.
n = 28: 28 snacks can be bought with 4 packs of 7 snacks.

Inductive Hypothesis: Suppose that $\mathsf{P}(24) \wedge \mathsf{P}(25) \wedge \cdots \wedge \mathsf{P}(k)$ is true for some arbitrary $k \geq 28$.

Inductive Step: We want to show that Robbie can buy exactly $k+1$ snacks. By the inductive hypothesis, we know that Robbie can buy exactly $k-4$ snacks, so he can buy another pack of 5 to get exactly $k+1$ snacks.

Conclusion: Therefore, $\mathsf{P}(n)$ holds for all integers $n \geq 24$ by the principle of strong induction.