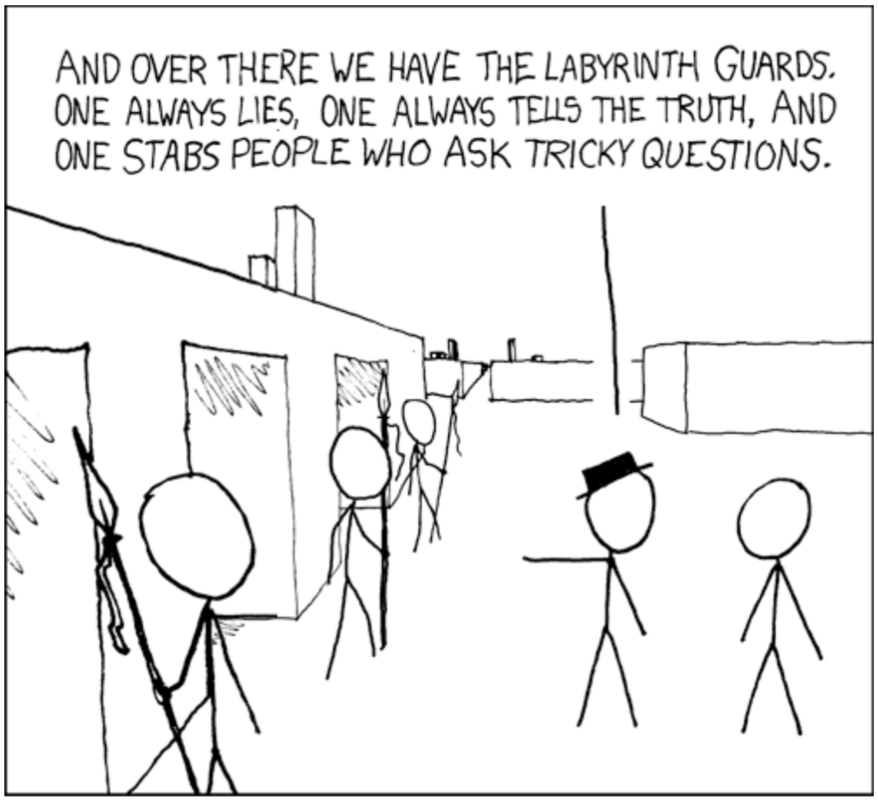
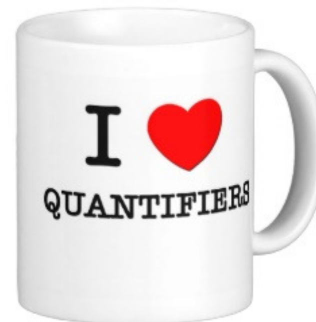
CSE 311: Foundations of Computing

Lecture 6: Predicate Logic, Logical Inference



Last Class: Quantifiers

We use *quantifiers* to talk about collections of objects.



$\forall x\ P(x)$

P(x) is true for every x in the domain

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x in the domain for which P(x) is true

read as "there exists x, P of x"

Last class: Predicate Logic to English (Natural)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x \, \exists y$ Greater(y, x)

For every positive integer, there is a larger positive integer.

$\exists y \, \forall x$ Greater(y, x)

There is a positive integer that is larger than every other positive integer.

$\forall x \, \exists y$ (Greater(y, x) $\wedge$ Prime(y))

For every positive integer, there is a prime that is larger.

Sound more natural without introducing variable names

Last class: English to Predicate Logic (Domain Restriction)

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"All red cats like tofu"

$$\forall x\, ((\mathrm{Red}(x) \land \mathrm{Cat}(x)) \rightarrow \mathrm{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\, ((\mathrm{Red}(y) \land \mathrm{Cat}(y)) \land \neg \mathrm{LikesTofu}(y))$$

Last class: Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= “x is a purple fruit”

(*) $\forall$x PurpleFruit(x) (“All fruits are purple”)

What is the negation of (*)?

(a) “there exists a purple fruit”

(b) “there exists a non-purple fruit”

(c) “all fruits are not purple”

Domain of Discourse
{plum, apple}

constant

(*) PurpleFruit(plum) $\wedge$ PurpleFruit(apple)

(a) PurpleFruit(plum) $\vee$ PurpleFruit(apple)

(b) $\neg$ PurpleFruit(plum) $\vee$ $\neg$ PurpleFruit(apple)

(c) $\neg$ PurpleFruit(plum) $\wedge$ $\neg$ PurpleFruit(apple)

Last class: De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Intuition: $\forall$ is like a giant AND over the domain
$\exists$ is like a giant OR over the domain

Last class: De Morgan's Laws for Quantifiers

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

These are equivalent but not equal

They have different English translations, e.g.:

There is no unicorn	$\neg\exists x$ Unicorn(x)
Every animal is not a unicorn	$\forall x \neg$ Unicorn(x)

Last class: De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

"There is no integer at least as large as every other integer"

$$\neg \exists x \forall y\ (x \geq y)$$
$$\equiv \forall x \neg \forall y\ (x \geq y)$$
$$\equiv \forall x\ \exists y \neg (x \geq y)$$
$$\equiv \forall x\ \exists y\ (y > x)$$

"For every integer, there is a larger integer"

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

"No even prime is greater than 2"

$$\neg \exists x\, (Even(x) \wedge Prime(x) \wedge Greater(x, 2))$$
$$\equiv \forall x\, \neg(Even(x) \wedge Prime(x) \wedge Greater(x, 2))$$
$$\equiv \forall x\, (\neg(Even(x) \wedge Prime(x)) \vee \neg Greater(x, 2))$$
$$\equiv \forall x\, ((Even(x) \wedge Prime(x)) \rightarrow \neg Greater(x, 2))$$
$$\equiv \forall x\, ((Even(x) \wedge Prime(x)) \rightarrow LessEq(x, 2))$$

"Every even prime is less than or equal to 2."

De Morgan's Laws for Quantifiers

We just saw that

$$\neg \exists x\, (P(x) \wedge R(x)) \equiv \forall x\, (P(x) \rightarrow \neg R(x))$$

Can similarly show that

$$\neg \forall x\, (P(x) \rightarrow R(x)) \equiv \exists x\, (P(x) \wedge \neg R(x))$$

De Morgan's Laws respect domain restrictions!
(It leaves them in place and only negates the other parts.)

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

Remain true when domain restrictions are used:

$$\neg \exists x\, (P(x) \wedge R(x)) \equiv \forall x\, (P(x) \rightarrow \neg R(x))$$
$$\neg \forall x\, (P(x) \rightarrow R(x)) \equiv \exists x\, (P(x) \wedge \neg R(x))$$

Scope of Quantifiers

$\exists x \; (P(x) \land Q(x))$ **vs.** $(\exists x \; P(x)) \land (\exists x \; Q(x))$

Scope of Quantifiers

$\exists x \ (P(x) \wedge Q(x))$ **vs.** $(\exists x \ P(x)) \wedge (\exists x \ Q(x))$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Scope of Quantifiers

Example: $\text{NotLargest}(x) \equiv \exists y \; \text{Greater}(y, x)$

$\equiv \exists z \; \text{Greater}(z, x)$

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\forall x \, (\exists y \, (P(x,y) \rightarrow \forall x \, Q(y, x)))$$

Quantifier "Style"

$$\forall x(\exists y\ (P(x,y) \rightarrow \forall\ x\ Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- Quantified variable names don't matter

$$\forall x\, \exists y\, P(x, y) \equiv \forall a\, \exists b\, P(a, b)$$

- Positions of quantifiers can sometimes change

$$\forall x\, (Q(x) \wedge \exists y\, P(x, y)) \equiv \forall x\, \exists y\, (Q(x) \wedge P(x, y))$$

- But: order is important...

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \mathrm{GreaterEq}(x, y)))$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "$x \geq y$"

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

"There is a number greater than or equal to all numbers."

$$\exists x \ \forall y \ \text{GreaterEq}(x, y)))$$

"Every number has a number greater than or equal to it."

$$\forall y \ \exists x \ \text{GreaterEq}(x, y)))$$

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= “x ≥ y”

“There is a number greater than or equal to all numbers.”

$\exists x\ \forall y$ GreaterEq(x, y)))

“Every number has a number greater than or equal to it.”

$\forall y\ \exists x$ GreaterEq(x, y)))

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Important: both include the case x = y

Different names does not imply different objects!

Quantification with Two Variables

	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

expression	when **true**	when **false**
$\forall x \forall y P(x, y)$ $\equiv \forall y \forall x P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$ $\equiv \exists y \exists x P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$ $\not\equiv$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

p	*q*	A	B
T	T	T	
T	F	T	
F	T	F	
F	F	F	

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

p	*q*	A	B
T	T	T	T
T	F	T	T
F	T	F	
F	F	F	

Given that A is true, we see that B is also true.

$$A \Rightarrow B$$

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where A is true:

p	*q*	A	B
T	T	T	T
T	F	T	T
F	T	F	?
F	F	F	?

When we zoom out, what have we proven?

New Perspective

Rather than comparing A and B as columns, zoom in on just the rows where B is true:

p	q	A	B	$A \to B$
T	T	T	T	T
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

When we zoom out, what have we proven?

$$(A \to B) \equiv \mathbf{T}$$

New Perspective

Equivalences

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ are the same

Inference

$A \Rightarrow B$ and $(A \rightarrow B) \equiv T$ are the same

Can do the inference by zooming in
to the rows where A is true

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with given facts (hypotheses)
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If A and $A \rightarrow B$ are both true, then B must be true
- Write this rule as $$\frac{A \,;\, A \rightarrow B}{\therefore\ B}$$
- Given:
  - If it is Friday, then you have a 311 class today.
  - It is Friday.
- Therefore, by Modus Ponens:
  - You have a 311 class today.

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	M.P. from 1, 2
5.	r	M.P. from 4, 3

Modus Ponens $\dfrac{A \, ; \, A \rightarrow B}{\therefore \; B}$

My First Proof!

Show that r follows from p, p → q, and q → r

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

Inference Rules

If A is true and B is true ....

Requirements: $\underline{A \; ; \; B}$

Conclusions: $\therefore C \; , \; D$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \; ; \; A \rightarrow B}{\therefore \quad B}$$

If I have A and $A \rightarrow B$ both true,
Then B must be true.

Axioms: Special inference rules

If I have nothing...

Requirements: __________

Conclusions: ∴ C , D

Then, C must be true

Then D must be true

Example (Excluded Middle):

$$\frac{}{\therefore\ A \lor \neg A}$$

A ∨¬A must be true.

Simple Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧

$$\frac{A \wedge B}{\therefore A,\ B}$$

Intro ∧

$$\frac{A\ ;\ B}{\therefore A \wedge B}$$

Elim ∨

$$\frac{A \vee B\ ;\ \neg A}{\therefore B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B,\ B \vee A}$$

Modus Ponens

$$\frac{A\ ;\ A \rightarrow B}{\therefore B}$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules