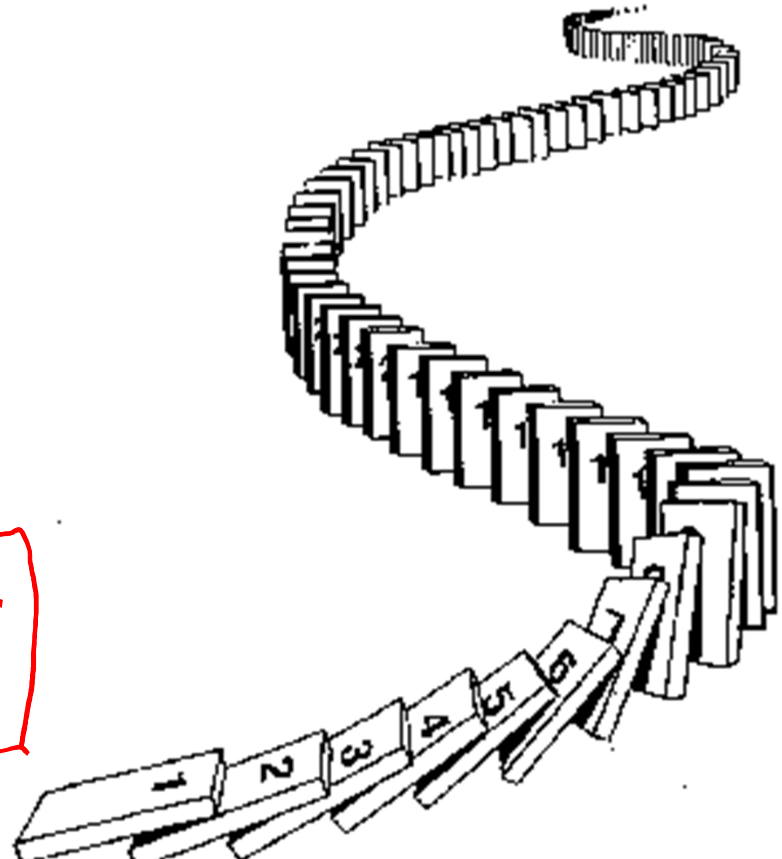
CSE 311: Foundations of Computing

Lecture 14: Induction

Read pinned Edstem post on backwards reasoning



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily

- **Particularly useful for reasoning about programs!**

```
for (int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

Prove $\forall a, b, m > 0\ \forall k \in \mathbb{N}\ ((a \equiv b\ (mod\ m)) \rightarrow (a^k \equiv b^k\ (mod\ m))$

Let $a, b, m > 0$ **be arbitrary. Let** $k \in \mathbb{N}$ **be arbitrary.**

Suppose that $a \equiv b \pmod{m}$**.**

We know that by multiplying congruences we get

$$\left(a \equiv b \pmod{m}\ \wedge\ a \equiv b \pmod{m}\right) \rightarrow a^2 \equiv b^2 \pmod{m}$$

Then, repeating this many times, we have:

$$\left(a^2 \equiv b^2 \pmod{m}\ \wedge\ a \equiv b \pmod{m}\right) \rightarrow a^3 \equiv b^3 \pmod{m}$$
$$\left(a^3 \equiv b^3 \pmod{m}\ \wedge\ a \equiv b \pmod{m}\right) \rightarrow a^4 \equiv b^4 \pmod{m}$$

...

$$\left(a^{k-1} \equiv b^{k-1} \pmod{m}\ \wedge\ a \equiv b \pmod{m}\right) \rightarrow a^k \equiv b^k \pmod{m}$$

The "…" is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$P(0)$;

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

Induction Is A Rule of Inference

Domain: Natural Numbers

1 $P(0)$

2 $\forall k\ (P(k) \longrightarrow P(k+1))$

$\therefore \forall n\ \ P(n)$

How do the givens prove P(5)?

P(1)? P(2)

1 P(0)

3. P(0) → P(1) Elim ∀

4 P(1) MP

5 P(1) → P(2) Elim ∀

P(2) MP

⋮

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\to P(1)} P(1) \xrightarrow{P(1)\to P(2)} P(2) \xrightarrow{P(2)\to P(3)} P(3) \xrightarrow{P(3)\to P(4)} P(4) \xrightarrow{P(4)\to P(5)} P(5)$$

First, we have P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
  Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
  Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.
...

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

1. $P(0)$

4. $\forall k\ (P(k) \to P(k+1))$
5. $\forall n\ P(n)$ By Induction from 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)

2. Let a be arbitrary

 3.1 P(a) Assumption

 ⋮

 3.100 P(a+1)

3. P(a) → P(a+1)

4. $\forall$k (P(k) $\longrightarrow$ P(k+1)) Intro $\forall$: 3

5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)
2. Let k be an arbitrary integer ≥ 0

3. $P(k) \rightarrow P(k+1)$
4. $\forall k\ (P(k) \rightarrow P(k+1))$ Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)	
2. Let k be an arbitrary integer ≥ 0	
3.1. P(k)	Assumption
3.2. ...	
3.3. P(k+1)	
3. $P(k) \rightarrow P(k+1)$	Direct Proof Rule
4. $\forall k\ (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5. $\forall n\ P(n)$	Induction: 1, 4

Translating to an English Proof

$$\frac{\begin{array}{c}P(0)\\ \forall k\ (P(k) \longrightarrow P(k+1))\end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0) **Base Case**

2. Let k be an arbitrary integer ≥ 0
   3.1. Suppose that P(k) is true

Inductive Hypothesis

   3.2. ...
   3.3. Prove P(k+1) is true

Inductive Step

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\ (P(k) \rightarrow P(k+1))$	Intro ∀: 2, 3
5.	$\forall n\ P(n)$	Induction: 1, 4

Conclusion

Translating to an English Proof

1. Prove P(0) **Base Case**
2. Let k be an arbitrary integer ≥ 0
   3.1. Assume that P(k) is true **Inductive Hypothesis**
   3.2. ...
   3.3. Prove P(k+1) is true **Inductive Step**
3. P(k) → P(k+1) Direct Proof Rule
4. ∀k (P(k) → P(k+1)) Intro ∀: 2, 3
5. ∀n P(n) Induction: 1, 4

Conclusion

Induction English Proof Template

[...Define P(n)...]

We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: *[...proof of $P(0)$ here...]*

Induction Hypothesis:

some

Suppose that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$.

Induction Step:

[...proof of $P(k+1)$ here...]

*The proof of $P(k+1)$ **must** invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof: Don't write ~~$P(n) =$~~

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \qquad\qquad\qquad\qquad = \; 1$ —
- $1 + 2 \qquad\qquad\qquad = \; 3$ —
- $1 + 2 + 4 \qquad\qquad = \; 7$ — $2^3 - 1$
- $1 + 2 + 4 + 8 \qquad = \; 15$ $2^{n+1} - 1$
- $1 + 2 + 4 + 8 + 16 = \; 31$

2^2 2^4 2^3 2^{n+1}

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$ $\forall n \in \mathbb{N}$

Proof 1. Let $P(n)$ be "$2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$".
We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

2. Base Case: LHS $2^0 = 1$ RHS $2^{0+1} - 1 = 2^1 - 1$

$2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ $\therefore 2^0 = 2^{0+1} - 1$ $= 2 - 1 = 1$

so $P(0)$ is true

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

3. Inductive Hypothesis: ~~Assume~~ Suppose that P(k) is true for some arbitrary $k \in \mathbb{N}$.
(that is: $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$)

4. Inductive Step: $\boxed{\text{Goal: Prove } P(k+1)}$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $\boxed{2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1.}$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

$$\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1}$$

$$2^0 + 2^1 + \cdots 2^k = 2^{k+1} - 1 \quad \text{by IH}$$

$$\therefore \underbrace{2^0 + 2^1 + \cdots + 2^k}_{} + 2^{k+1} = \underbrace{2^{k+1} + 2^{k+1}}_{} - 1$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be "**$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$**". We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

   $$2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1 \quad \textbf{by IH}$$

   Adding 2^{k+1} **to both sides, we get:**

   $$2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

   So, we have $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$**, which is exactly** P(k+1)**.**

5. Conclusion: ∴ P(n) is true for all $n \in \mathbb{N}$

   $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

   We can calculate

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0 + 2^1 + \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} && \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1)**.**

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

   We can calculate

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} && \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1).

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

O+

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers** n **by induction.**

2. Base Case: LS = 0 + … + 0 = 0
   RS = 0(0+1)/2 = 0
   ∴ P(0) is true

$\sum_{i=0}^{0} i = 0$

Summation Notation

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + 3 + \ldots + n$$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

Summation Notation

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + 3 + \ldots + n$$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + ... + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + ...+ k = k(k+1)/2

0 +

Induction Step:

Goal: Prove P(k+1): $0+1+\cdots+k+k+1 = \frac{(k+1)(k+2)}{2}$

"some" or "an" ✗
not any!

$0+1+\cdots+k+k+1$

$= \frac{k(k+1)}{2} + k+1$ by I.H

$= (k+1)\left[\frac{k}{2}+1\right]$

$= (k+1)\left(\frac{k+2}{2}\right)$ ∴ P(k+1) follows

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + …+ k = k(k+1)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** 1 + 2 + …+ k = k(k+1)/2
4. **Induction Step:**

   Goal: Show P(k+1)**, i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + …+ k = k(k+1)/2
4. **Induction Step:**

$$\begin{aligned} \underline{1 + 2 + \ldots + k} + (k+1) &= (1 + 2 + \ldots + k) + (k+1) \\ &= k(k+1)/2 + (k+1) \quad \textbf{by IH} \\ &= (k+1)(k/2 + 1) \\ &= (k+1)(k+2)/2 \end{aligned}$$

So, we have shown 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2, **which is exactly** P(<u>k+1</u>).

5. **Thus** P(n) **is true for all** n ∈ℕ, **by induction.**

never
"any"
~~means~~
"every"
"for all"

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?

- Define predicate $Q(k) = P(k + b)$ for all k.

  $\in \mathbb{N}$

  – Then $\forall n\ Q(n) \equiv \forall n \geq b\ \ P(n)$

- Ordinary induction for Q:
  – Prove $Q(0) \equiv P(b)$
  – Prove

$$\forall k\ \big(Q(k) \longrightarrow Q(k+1)\big) \equiv \forall k \geq b\big(P(k) \longrightarrow P(k+1)\big)$$

Inductive Proofs starting at $\boldsymbol{b}$ in 5 Easy Steps

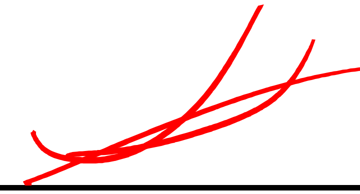
1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$



Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**

2. Base Case ($n=2$): $3^2 = 9$ $2^2+3 = 7$

$\therefore 3^2 \geq 2^2+3$ $\therefore$ P(2) is true

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

3 Ind Hyp

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** “$3^n \geq n^2+3$”. **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$. **I.e., suppose** $3^k \geq k^2+3$.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**. I.e., suppose** $\underline{3^k} \geq k^2+3$**.**
4. **Inductive Step:**

$$\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3^{k+1} \geq (k+1)^2+3} \quad = k^2$$

$$3^{k+1} = 3(3^k)$$

$$\geq 3(k^2+3) \quad \text{by I.H.}$$

$$= 3k^2+9$$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers $n \geq 2$ by induction.
2. Base Case (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so P(2) is true.
3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 2$. I.e., suppose $3^k \geq k^2+3$.
4. Inductive Step:

   Goal: Show P(k+1), i.e. show $3^{k+1} \geq \underline{(k+1)^2+3} = \underline{k^2+2k+4}$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** “$3^n \geq n^2+3$”. **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$. **I.e., suppose** $3^k \geq k^2+3$.
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

$$
\begin{aligned}
3^{k+1} &= 3(3^k) \\
&\geq 3(k^2+3) \text{ by the IH} \\
&= 3k^2+9 \\
&= k^2+2k^2+9 \geq k^2+2k+4 = (k+1)^2+3 \\
&\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 2.
\end{aligned}
$$

Therefore P(k+1) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$. **I.e., suppose** $3^k \geq k^2+3$.
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3^{k+1} \geq (k+1)^2+3=k^2+2k+4}$

$$
\begin{aligned}
3^{k+1} &= 3(3^k) \\
&\geq 3(k^2+3) \text{ by the IH} \\
&= k^2+2k^2+9 \\
&\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 2.
\end{aligned}
$$

   Therefore P(k+1) **is true.**

5. **Thus** P(n) **is true for all integers** $n \geq 2$, **by induction.**