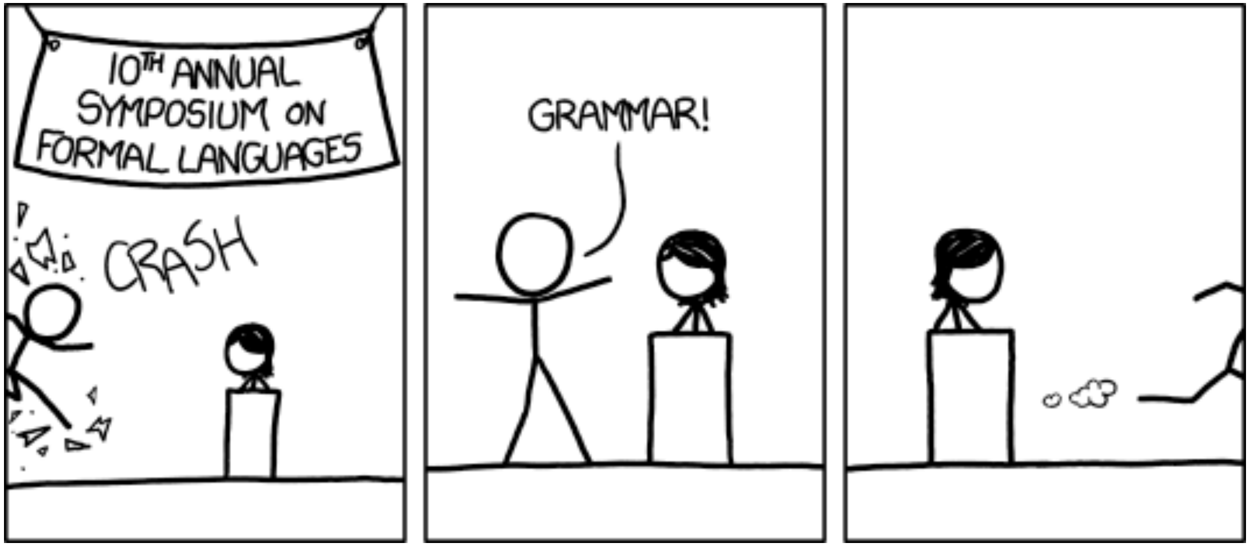
CSE 311: Foundations of Computing

Lecture 19: Context-Free Grammars



[Audience looks around]
“What is going on? There must be some context we’re missing”

Last class: Languages: Sets of Strings

- Subsets of strings are called *languages*
- Examples:
  - Σ^* = All strings over alphabet Σ
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Binary strings with an equal # of 0's and 1's
  - Legal variable names in Java/C/C++
  - Syntactically correct Java/C/C++ programs
  - Valid English sentences

Last class: Regular Expressions

Regular expressions over Σ

- **Basis:**

  ε is a regular expression (could also include $\varnothing$)

  a is a regular expression for any $a \in \Sigma$

- **Recursive step:**

  If **A** and **B** are regular expressions then so are:

  A $\cup$ **B**

  AB

  A*

Last class: Regular Expression is a "pattern"

ε matches the **empty string**

a matches the one character string *a*

A ∪ **B** matches all strings that either **A** matches or **B** matches (or both)

AB matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another

Yields a *language* = the set of strings matched by the regular expression

Last class: Examples

A ∪ B

$0^* 1^* \cup 1^*$

S matches A and S matches

Regular Expression	Language
001*	{00, 001, 0011, 00111, …}
0*1* 00111 11 0	{Binary strings with any number of 0s followed by any number of 1s}
(0 ∪ 1) 0 (0 ∪ 1) 0	{0000, 1000, 0010, 1010}
(0*1*)*	{All binary strings}={0,1}*
(0 ∪ 1)* Σ^*	{All binary strings}={0,1}*
(0 ∪ 1)* 0110 (0 ∪ 1)*	{All binary strings containing substring 0110}

Regular Expressions in Practice

- Used to define the *tokens* of a programming language
  - legal variable names, keywords, etc.
- Used in `grep`, a program that does pattern matching searches in UNIX/LINUX
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

```
Pattern p = Pattern.compile("a*b");
Matcher m = p.matcher("aaaaab");
boolean b = m.matches();
```

`[01]`	a 0 or a 1	`^` start of string	`$` end of string
`[0-9]`	any single digit	`\.` period `\,` comma	`\-` minus
`.`	any single character		
ab	a followed by b	(AB)	
`(a|b)`	a or b	$(A \cup B)$	
a`?`	zero or one of a	$(A \cup \varepsilon)$	
a`*`	zero or more of a	A^*	
a`+`	one or more of a	AA^*	

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)

Examples

$\Sigma = \{0, 1\}$

0 00 0...

- **All binary strings that have an even # of 1's**

✗ $(0^* 1 0^* 1 0^*)^*$ 000

001

√ $(0 \cup (10^*1))^*$

✗ $(0 \cup 11)^*$ 101

1010101

✗ $0^*(10^*1)^* 0^*$

✗ $(10^*1)^*$

✗ $(0 \cup 11)^* \cup 0^*$ 101

✗ $(0^* 1 0^* 1)^*$ 000

$a + (bx)$

Examples

- All binary strings that have an even # of 1's

e.g., $0^* (10^*10^*)^*$

Examples

- All binary strings that have an even # of 1's

  e.g., ***0* (10*10*)****

$\Sigma^* 101 \Sigma^*$

- All binary strings that *don't* contain 101

101 X $0^*(00^* \cup 1)^*$

110 X $000^*(100^*1)^*$

01 X $(00 \cup 1)^*$

010 $0^*(1^*000^*)^*$ X

10 $0^*((00^*0) \cup 1)^*$

01 $(00 \cup 1)^* 0^*$ X

10001 $0^*(00 \cup 1)^* 0^*$ X

Examples

- All binary strings that have an even # of 1's

  e.g., $0^* (10^*10^*)^*$

- All binary strings that *don't* contain 101

  e.g., $0^* (1 \cup 000^*)^* \, 0^*$

  at least two 0s between 1s

Limitations of Regular Expressions

- Not all languages can be specified by regular expressions
- Even some easy things like
  - Palindromes
  - Strings with equal number of 0's and 1's
- But also more complicated structures in programming languages
  - Matched parentheses
  - Properly formed arithmetic expressions
  - etc.

Context-Free Grammars

- A Context-Free Grammar (CFG) is given by a finite set of substitution rules involving
  - Alphabet Σ of *terminal symbols* that can't be replaced
  - A finite set **V** of *variables* that can be replaced
  - One variable, usually **S**, is called the *start symbol*

- The substitution rules involving a variable **A**, written as

$$\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$$

where each w_i is a string of variables and terminals
  – that is $w_i \in (\mathbf{V} \cup \Sigma)^*$

How CFGs generate strings

- Begin with "**S**"

- If there is some variable **A** in the current string, you can replace it by one of the w's in the rules for **A**
  - $\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$
  - Write this as $x\mathbf{A}y \Rightarrow xwy$
  - Repeat until no variables left

- The set of strings the CFG describes are all strings, containing no variables, that can be *generated* in this manner after a finite number of steps

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

$S \Rightarrow 0S \Rightarrow 0S1 \Rightarrow 0\varepsilon 1 = 01$

$S \Rightarrow 0S \Rightarrow 00S \Rightarrow 00$

Example Context-Free Grammars

Example: $\mathbf{S} \to 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

0*1*

any # of 0s followed by
any # of 1s

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

0*1*

Example: $\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$

$S \Rightarrow 0S0 \Rightarrow 00S00 \Rightarrow 001S100$

$\Rightarrow 0011100$

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

0*1*

Example: $\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$

The set of all binary palindromes

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

n 00111 y

(i.e., matching *0*1 but with same number of 0's and 1's)**

$S \to 0S1 \mid \varepsilon$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1** but with same number of 0's and 1's)

$$\mathbf{S} \rightarrow 0\mathbf{S}1 \mid \varepsilon$$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1 but with same number of 0's and 1's)**

$$\mathbf{S} \rightarrow 0\mathbf{S}1 \mid \varepsilon$$

Grammar for $\{0^n 1^{2n} : n \geq 0\}$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1 but with same number of 0's and 1's)**

$$\mathbf{S} \rightarrow 0\mathbf{S}1 \mid \varepsilon$$

Grammar for $\{0^n 1^{2n} : n \geq 0\}$

$$\mathbf{S} \rightarrow 0\mathbf{S}11 \mid \varepsilon$$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1** but with same number of 0's and 1's)

$$S \rightarrow 0S1 \mid \varepsilon$$

Grammar for $\{0^n 1^{n+1} 0 : n \geq 0\}$

$\underbrace{0^n 1^n}\ \underbrace{10}$

$S \rightarrow AB$

$A \rightarrow 0A1 \mid \varepsilon$

$B \rightarrow 10$

$S \rightarrow A10$

$A \rightarrow 0A1 \mid \varepsilon$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching *0*1** but with same number of 0's and 1's)

S $\rightarrow$ 0**S**1 | ε

Grammar for $\{0^n 1^{n+1} 0 : n \geq 0\}$

S $\rightarrow$ A10

A $\rightarrow$ 0**A**1 | ε

Example Context-Free Grammars

$\Sigma = \{ (,) \}$

Example: $\mathbf{S} \rightarrow \mathbf{(S)} \mid \mathbf{SS} \mid \varepsilon$

$S \Rightarrow (S) \Rightarrow (SS) \Rightarrow ((S)S)$

$\Rightarrow ((S)(S)) \Rightarrow (()(S)) \Rightarrow (()())$

Example Context-Free Grammars

Example: $S \to (S) \mid SS \mid \varepsilon$

The set of all strings of matched parentheses

)

)(

Example Context-Free Grammars

Binary strings with equal numbers of 0s and 1s

(not just 0^n1^n, also 0101, 0110, etc.)

$S \to 0S1 \mid 1S0 \mid \varepsilon \mid SS$

0110

Example Context-Free Grammars

Binary strings with equal numbers of 0s and 1s

(not just 0^n1^n, also 0101, 0110, etc.)

$$S \rightarrow SS \mid 0S1 \mid 1S0 \mid \varepsilon$$

An easy structural induction can show that everything generated by S has an equal # of 0s and 1s

Why does this generate all such strings?